International Journal of Information Technology and Knowledge Management, 2012, Volume 6, No. 1, pp. 86-92

On Complete Metric Space of Common Fixed Point Theorem

Rajesh Shrivasatava
Govt. Beneezer Science and Commerce College, Bhopal (M.P.), India

Abstract: In this present paper, we apply some complete metric space of common fixed point theorem using four mapping. These results on ordinary metric space.
Keywords: Weakly compatible mappings, Common Fixed Point Theorem.

Introduction: In 1998, Jungck and Rhoades [5] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but not conversing Brain fisher proved an important common fixed point theorem. We introduce a binary operation definition on ordinary metric spaces and their some properties on operation metric space we use four mappings in complete metric space on common fixed point theorem. This present fixed result for two pair a maps satisfying a new condition by using concept weakly compatible maps in a complete metric space.

2.Preliminaries:

Definition 2.1: Let $*: R^+ \times R^+ \to R^+$ be a binary operation satisfying the following condition:

1) * is associative and commutative

2) * is continuous

Examples: a * b = max {a, b}, a * b = a + b, a * b = ab a * b = ab + a + b and

$$a * b = \frac{ab}{\max\{a, b, 1\}} \text{ for each } a, b \in R^+$$

Definition 2.2.: The binary operation is said to satisfy α- property if. There exists a positive real number α such that

$$a * b \le \alpha \max\{a, b\}$$

for all $a, b \in R^+$

Example: Suppose a * b * c = a + b + c for all $a, b, c \in R^+$ then for $\alpha \ge 2$ we have $a * b * c \le \alpha \max\{a, b, c\}$

Definition 2.3: Let A and S be mappings from a metric space (X, d) into itself. A and S are said to be weakly compatible if they commute at their coincidence points, that is, Ax = Sx for some $x \in X$ implies that ASx = Sax.

3. Main Results

Theorem 3.1: Let (X,d) be a complete metric space such that * satisfies α-property with $\alpha > 0$, Let A, B, S, and T be self mappings of X in to itself satisfying the following conditions.

1) $A(x) \subseteq T(x)$, $B(x) \subset S(x)$ and T(x) or S(x) is a closed subset of X.

2) The pairs (A . S) and (B . T) are weakly compatible

For all $x, y \in X$

3) $$d(Ax,By,t) \le K_1 \left(d\left(Sx, Ty, t\right) * d\left(Ax, Sx, t\right)\right)$$

$$+ K_2 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, Ty, t\right) + d\left(By, Ty, t\right)}{2}$$

$$+ K_3 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, By, t\right) + d\left(Ax, Ty, t\right)}{2}$$

$$+ K_4 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, Bx, t\right) + d\left(Bx, Tx, t\right)}{2}$$

Where $K_1, K_2, K_3, K_4 > 0$, $t > 0$, and $0 < \alpha (K_1 + K_2 + K_3 + K_4) < 1$. Then A,B, S and T have a unique common fixed point in X.

Proof: Let x_0 be an arbitrary point in X . By (i), we can defined Inductively a Sequence $\{ y_n \}$ in X.

$\Rightarrow y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$ for n = 0, 1, 2………

We claim that the sequence $\{y_n\}$ is a Cauchy Sequence.

Using (iii), we have

$$d(y_{2n}, y_{2n+1}, t) = d\,(Ax_{2n}, bx_{2n+1}, t)$$

$$\le K_1 (d\,(Sx_{2n}, Tx_{2n+1}, t) * d(Ax_{2n}, Sx_{2n}, t)$$

$$+ K_2 \left(d\left(Sx_{2n}, Tx_{2n+1}, t\right)\right) * \frac{d\left(Sx_{2n}, Tx_{2n+1}, t\right) + d\left(Bx_{2n+1}, Tx_{2n+1}, t\right)}{2}$$

$$+ K_3 \left(d\left(Sx_{2n}, Tx_{2n+1}, t\right)\right) * \frac{d\left(Sx_{2n}, Bx_{2n+1}, t\right) + d\left(Ax_{2n}, Tx_{2n+1}, t\right)}{2}$$

$$+ K_4 \left(d\left(Sx_{2n}, Tx_{2n+1}, t\right)\right) * \frac{d\left(Sx_{2n}, Bx_{2n}, t\right) + d\left(Bx_{2n}, Tx_{2n}, t\right)}{2}$$

$$= K_1 (d\,(y_{2n-1}, y_{2n}, t) * d(y_{2n}, y_{2n-1}, t)$$

$$+ K_2 \left(dy_{2n-1}\ y_{2n}, t\right) * \frac{d\left(dy_{2n-1}, y_{2n}, t\right) + d\left(y_{2n+1}, y_{2n}, t\right)}{2}$$

$$+ K_3 \left(d\left(y_{2n-1}\ y_{2n}, t\right)\right) * \frac{d\left(y_{2n-1}, y_{2n+1}, t\right) + d\left(y_{2n}, y_{2n}, t\right)}{2}$$

$$+ K_4 \left(d\left(y_{2n-1}\ y_{2n}, t\right)\right) * \frac{d\left(y_{2n-1}, y_{2n+1}, t\right) + d\left(y_{2n}, y_{2n+1}, t\right)}{2}$$

Set $d_n = d(y_n, y_{n+1})$ using the above inequality we get

$$d_{2n} \leq K_1 (d_{2n-1} * d_{2n-1}, t) + K_2\left(d_{2n-1} * \frac{d_{2n-1} + d_{2n}, t}{2}\right)$$

$$+ K_3\left(d_{2n-1} * \frac{d_{2n-1} + d_{2n}, t}{2}\right)$$

$$+ K_4\left(d_{2n-1} * \frac{d_{2n-1} + d_{2n}, t}{2}\right)$$

Hence

$$d_{2n} \leq K_1\ \alpha d_{2n-1}, + K_2\alpha \max\left\{d_{2n-1} * \frac{d_{2n-1} + d_{2n}}{2}\right\} + K_3\ \alpha \max\left\{d_{2n-1} * \frac{d_{2n-1} + d_{2n}}{2}\right\}$$

$$+ K_4\ \alpha \max\left(d_{2n-1} * \frac{d_{2n-1} + d_{2n}, t}{2}\right)$$

Suppose $d_{2n} > d_{2n-1}$, we obtain

$$d_{2n} \leq K_1\ \alpha\ d_{2n} + K_2\ \alpha\ d_{2n}, + K_3\ \alpha\ d_{2n} + K_4 \alpha\ d_{2n} < d_{2n}$$

Which is a contradiction, Hence $d_{2n} \leq d_{2n-1}$ Similarly it is easy to see that $d_{2n+1} \leq d_{2n}$

$\therefore\ d_n \leq d_{n-1}$ for n = 1, 2, …….

Using the above inequality

$$dn \leq \alpha\ (K_1 + K_2 + K_3 + K_4)\ d_{n-1} = kd_{n-1}$$

Where $\alpha(K_1 + K_2 + K_3 + K_4) = K < 1$ So

$$d_n \leq Kd_{n-1} \leq K^2 d_{n-2} \leq \ldots\ldots\ldots\ldots \leq K^n d_0$$

$\therefore\ d(y_n, y_{n+1}) \leq K^n (y_0, y_1) \rightarrow 0$ as $n \rightarrow \infty$

Suppose m > n

$$d(y_n . y_m) \leq d (y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \ldots\ldots\ldots + d(y_{m-1}, y_m)$$

$$\leq K^n d(y_0, y_1) + k^{n+1} (y_0, y_1) + \ldots\ldots\ldots + k^{n-1} d(y_0, y_1)$$

$$= \frac{K^n}{1-k} d\left(y_0 . y_1\right) \text{ As } n, m \rightarrow \infty$$

It follows that the sequence $\{y_n\}$ is a Cauchy Sequence and by the completeness of X.

$\{y_n\}$ converges to $y \in x$.

$$\therefore \lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = \lim_{n\to\infty} Tx_{2n+1} = y$$

Assume that T(x) is a closed subset of x . $\exists\, v \in x \Rightarrow Tv = y$

Suppose $Bv \neq y$ their by (iii) we obtain

$d(Ax_{2n}, Bv, t) \leq K_1 (d\,(Sx_{2n}, Tv, T) * d(Ax_{2n}, Sx_{2n}, t))$

$$+ K_2\,(d(Sx_{2n}, Tv, t)) * \frac{d(Sx_{2n}, Tv, t) + d(Bv, Tv, t)}{2}$$

$$+ K_3\,(d(Sx_{2n}, Tv, t)) * \frac{d(Sx_{2n}, Bv, t) + d(Ax_{2n}, Tv, t)}{2}$$

$$+ K_4\,(d(Sx_{2n}, Tv, t)) * \frac{d(Sx_{2n}, Bx_{2n}, t) + d(Bx_{2n}, Tx_{2n}, t)}{2}$$

$$d(y, Bv, t) \leq K_1\,(d\,(y, Tv, t) * d(yy, t) + K_2\,(d\,(y, Tv, t))) * \frac{d(y, Tv, t) + d(Bv, Tv, t)}{2}$$

$$+ K_3\,(d\,(y, Tv, t)) * \frac{d(y, Bv, t) + d(y, Tv, t)}{2}$$

$$+ K_4\,(y, Tv, t) * \frac{d(y, Bv, t) + d(y, Tv, t)}{2}$$

$$\leq K_1\,\alpha \max\{d(y, Tv, t) = 0\} + K_2\,\alpha \max\left\{o, \frac{o + d(y, Tv, t)}{2}\right\} + K_3\,\alpha \max\left\{o, \frac{d(y, Bv, t) + o}{2}\right\}$$

$$+ K_4\,\alpha \max\left\{o, \frac{d(y, Bv, t) + o}{2}\right\}$$

$\leq d(y, Bv)$

It follows that Bv = y = Tv Since B and T are weakly compatible, we Have BTv = TBv And so By = Ty

Suppose $y \neq By$, by (iii)

$$\lim_{n\to\infty} d(Ax_{2n}, By, t) \leq \lim_{n\to\infty} [K_1\,(d\,(Sx_{2n}, Ty, t)) * d(Ax_{2n}, Sx_{2n}, t)]$$

$$+ K_2\,(d(Sx_{2n}, Ty, t)) * \frac{d(Sx_{2n}, Ty, t) + d(By, Ty, t)}{2}$$

$$+ K_3\,(d(S_{2n}, Ty, t)) * \frac{d(Sx_{2n}, By, t) + d(Ax_{2n}, Ty, t)}{2}$$

$$+ K_4 \left(d(Sx_{2n}, Ty, t)\right) * \frac{d(Sx_{2n}, Bx_{2n}, t) + d(Bx_{2n}, Tx_{2n}, t)}{2}$$

Hence

$$d(y, By, t) \leq K_1 \left(y, Ty, t\right) * d(y, y, t) + K_2\left(d\left(y, Ty, t\right)\right) * \frac{d(y, ty, t) + d(By, Ty, t)}{2}$$

$$\leq K_3 \left(d(y, Ty, t)\right) * \frac{d(y, By, t) + d(y, Ty, t)}{2}$$

$$\leq K_4 \left(d(y, Ty, t)\right) * \frac{d(y, y, t) + d(y, Ty, t)}{2}$$

$$\leq K_1 \ \alpha \max \left\{d(y, Ty, t), d(y, y, t)\right\} + K_2 \ \alpha \max\left(d\left(y, Ty, t\right)\right) * \frac{d(y, ty, t) + d(By, Ty, t)}{2}$$

$$+ K_3 \ \alpha \max\left(d\left(y, Ty, t\right)\right) * \frac{d(y, By, t) + d(y, Ty, t)}{2}$$

$$+ K_4 \ \alpha \max\left(d\left(y, Ty, t\right)\right) * \frac{d(y, y, t) + d(y, Ty, t)}{2}$$

$<d$ (y, By) and So By = y Since $B(x) \subseteq S(x)$, there exists $w \in x$ such that Sw = y It $Aw \neq y$ by (iii) we have

$d(Aw, By, t) \leq K_1 (d (Sw, Ty, t) * d(Aw, Sw, t))$

$$+ K_2 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, Ty, t) + d(By, Ty, t)}{2}$$

$$+ K_3 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, By, t) + d(Aw, Ty, t)}{2}$$

$$+ K_4 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, Bw, t) + d(Bw, Tw, t)}{2}$$

And it follows that

$d(Aw, y, t) \leq K_1 (d (Sw, y, t) * d(Aw, Sw, t))$

$$+ K_2 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, y, t) + d(Ay, y, t)}{2}$$

$$+ K_3 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, y, t) + d(Aw, y, t)}{2}$$

$$+ K_4 \left(d(Sw, Ty, t)\right) * \frac{d(Sw, Bw, t) + d(Bw, Tw, t)}{2}$$

$< d\ (Aw, y, t)$

$\Rightarrow$ Aw = y and hence Aw = Aw = y

Suppose Ay $\neq$ y Then by (iii)

$d\ (Ay, y) = d\ (Ay, By)$

$\leq K_1\ (d\ (sy, Ty, t) * d\ (Ay, Sy, t)$

$$+ K_2\ (d(Sy, Ty, t)) * \frac{d(Sy, Ty, t) + d(By, Ty, t)}{2}$$

$$+ K_3\ (d(Sy, Ty, t)) * \frac{d(Sy, By, t) + d(Ay, Ty, t)}{2}$$

$$+ K_4\ (d(Sy, Ty, t)) * \frac{d(Sy, By, t) + d(Ay, Ty, t)}{2}$$

$$= K_1\ (d\ (Sy, y, t)) * d(Ay, Sy, t) + K_2\ (Sy, y, t)\frac{d(Sy, y, t) + (Ay, y, t)}{2}$$

$$+ K_3\ (d(Sy, y, t)) * \frac{d(Sy, y, t) + (Ay, y, t)}{2}$$

$$+ K_4\ (d(Sy, y, t)) * \frac{d(Sy, y, t) + d\ (By, y, t)}{2}$$

$$= K_1\ \alpha \max\ (d(Sy, y, t)) * (Ay, Sy, t)$$

$$+ K_2\ \alpha \max\ (d(Sy, y, t)) * \frac{d(Sy, y, t) + B(Ay, y, t)}{2}$$

$$+ K_3\ \alpha \max\ (d(Sy, y, t)) * \frac{d(Sy, y, t) + (Ay, y, t)}{2}$$

$$+ K_4\ \alpha \max\ (d(Sy, y, t)) * \frac{d(Sy, y, t) + (By, y, t)}{2}$$

$< d(Ay, y, t)$

And So Ay = y Thus Ay = Sy = By = Ty = y

$\therefore$ y is common fixed point for A, B, S and T the proof is similar when S(x) is assumed to be a closed subset A x

The uniqueness s at y follow from (III)

The uniqueness of y follows form (iii)

Corollary:

Let (x, d) be a complete metric space, Let A, B, S and T be self mappings of x into itself satisfying the following conditions.

1) $A(x) \subseteq T(x)$, $B(x) \subset S(x)$ and T(x) or S(x) is closed subset of x.

2) The pairs (A, S) and (B, T) weakly compatible $\forall$ x, y, $\in$ X.

3) 3)

$$d(Ax,By,t) \le K_1 \left(d\left(Sx, Ty, t\right) * d\left(Ax, Sx, t\right)\right)$$

$$+ K_2 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, Ty, t\right) + d\left(By, Ty, t\right)}{2}$$

$$+ K_3 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, By, t\right) + d\left(Ax, Ty, t\right)}{2}$$

$$+ K_4 \left(d\left(Sx, Ty, t\right)\right) * \frac{d\left(Sx, Bx, t\right) + d\left(Bx, Tx, t\right)}{2}$$

Where $K_1, K_2, K_3, K_4 > 0$ and $0 < K_1 + K_2 + K_3 + K_4 < ½$ then A, B S, and T have a unique common fixed point in X.

References:

1) M.A. Ahmed, Common fixed point theorems for weakly compatible mappings, Rocky Mountain J. Math. 33 (4) (2003), 1189 – 1203.
2) R. Chugh and S. Kumar, Common fixed points for weakly compatible maps, Proc. Indian Acad. Sci. Math. Sci. 111 (2) (2001), 241-247.
3) Lj. B. ciric and J.S. Ume, Some common fixed point theorems for weakly compatible mappings, J. Math. Anal. Appl. 314 (2) (2006), 488-499.
4) G. Jungck, Common fixed points for non-continuous non-self maps on non metric spaces, Far East J. Math. Sci. 4 (2) (1996), 199-215.
5) G. Jungck and B. E. Rhoades, Fixed point for set valued functions without continuity.
6) V. Popa, A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation, Filomat 19 (2005), 45-51.