Improved Fractional Digital Delay Filter

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Abstract: In wide application of communication, audio and music technology, speech coding, antenna and time delay estimation. The sampling instant provides the sampled frequency at regular interval of time. Digital delay filters provide the fine tuning of sampling interval which is implemented for band limited interpolation. In this paper, Design technique of fractional delay filter and application is given.

1. INTRODUCTION

According to the sampling theorem, the sampling rate must satisfy Nyquist criterion, for a sample set to represent the original continuous signal, that is, \( f_s \geq 2f_m \) where, \( f_s \) is the sampling frequency and \( f_m \) be the continuous frequency. However the sampling rate is not sufficient for many applications because the sampling instant must be selected properly. For example, in digital communications, the received bit or symbol are made based on sample of received continuous time pulse which should be taken between each pulse to minimize probability of error. This requires the synchronization between sampling instant and sampling frequency.

Another problem is modeling of musical instrument which is characterized by differential equation, a physical system producing acoustical vibrations, propagation delays due to a finite speed of vibration in strings, tubes must be simulated accuracy. Otherwise the instrument will sound out of tone. The accurate playing of instrument requires changing of systems parameters.

Fig. 1: For \( D = 3 \)

These two examples are typical application of Fractional delay filters. In uniform sampling is used here, then interpolation is applied between two samples [1], [2], [3]. The computation became very large due to large computation, system becomes very slow. Hence fraction delay filter is used. The fractional delay means, assuming uniform sampling a delay that is a non uniform integer multiple of the sampling interval. Using fractional delay filters developed for uniform sampled Signal and yet the location between the samples.

Fig. 2: For \( D = 3.3 \)

In this paper we review the principle and max flat of fractional delay digital filter first we introduce the digital fractional delay problem and max flat and compare with known techniques for designing of recessive (IIR Filter). A special case of a small variation delay (\( D \leq 1 \)) is considered.

2. DISCRETE TIME SYSTEM FOR IDEAL DELAY

The delay system must be reduced band limited using low pass filter while delay mainly shifts the impulse response in time domain. This impulses response of fraction delay filter is shifted and sampled sine fraction. As shown in Fig. (1)

We want to express the discrete time version of the delay operation for sample band limited signal. The response of the system is

\[
y[n] = x(n-D)
\]

where

\[
D = \frac{\zeta}{T}
\]

Note that \( \frac{\zeta}{T} \) generally irrational since \( \zeta \) is not an integer multiple of sampling interval \( T \). Unfortunately \( \zeta \) is an integer multiple of sampling interval \( T \). The output response \( y[n] \) are equal to delayed samples of the input sequence \( x[n] \) and delay element is called digital delay line [4].
The spectrum of input sequence

\[ X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad |\omega| \leq \pi \]  

\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n] e^{-j\omega n} \]  

\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n-D]e^{-j\omega n} = e^{-j\omega D} X(e^{j\omega}) \]  

Where \( m \) be any integer value.

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D} \]  

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = e^{-j\omega D} \quad |\omega| \leq \pi \]  

In Z-transform form terms

\[ H_d(z) = \frac{Y(z)}{X(z)} = z^{-D} \]  

Now, \( D \) may written as

\[ D = D_{int} - d \]  

Where \( 0 \leq d \leq 1 \) the fractional delay and integer part of is \( D_{int} \) is given by

\[ D_{int} = \lfloor D \rfloor \]  

Where \( \lfloor . \rfloor \) is greatest integer function. It is called floor function and defined by

\[ \lfloor x \rfloor = \max k \text{ Where } k \leq x \]

To understand how to produce a fractional delay using a discrete time system it is necessary to discuss interpolation technique. Interpolation of discrete time signal is based on the fact that the amplitude of the corresponding continuous time band limited signal change smoothly between the sampling instant.

Now we known that from reconstruction formula of the sampled signal

\[ x_c(t) = \sum_{n=-\infty}^{\infty} x[nT] \sin \left( \frac{\omega}{2}(t-nT) \right) \]  

\[ x_c(t) = \sum_{n=-\infty}^{\infty} x[nT] \sin \left( \frac{\omega}{2\pi}(t-nT) \right) \]

The equation becomes

\[ y[n] = x(n-D) = \sum_{k=-\infty}^{\infty} x[k] \sin c(n-D-k) \]  

3. DESIGN METHOD FOR FRACTIONAL DELAY FIR FILTER

The Transfer function of FIR filter is

\[ H(z) = \sum_{n=0}^{N} h[n]z^{-n} \]  

Where \( N \) is the order of filter and \( h(n) = \{0, 1\ldots, N\} \) are real co-efficient that form the impulse response of FIR Filter.

Length of the filter

\[ L = N + 1 \]  

In the design procedure our aim is to minimize the error function defined by

\[ E(e^{j\omega}) = H(e^{j\omega}) - H_d(e^{j\omega}) \]  

\( i.e. \), The difference of the ideal frequency response and approximation. The impulse response \( h[n] \) of the least square fractional delay FIR filter can be expressed as

\[ H(n) = \begin{cases} \sin c(n-D) & 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases} \]

The delay should be located between the central taps of the filter when \( N \) is odd \( (L = N + 1 \) is even) or with in half a sample from the central tap when \( N \) is even \( (L \) is odd).

Since then the approximation error is smallest. This means that the delay \( D \) should be chosen so that the following inequality is valid.

\[ \frac{N-1}{2} \leq D \leq \frac{N+1}{2} \]  

When \( N \) is even \( (L \) is odd) the integer part of the delay should be chosen so that

\[ D_{int} = \begin{cases} \frac{N}{2} & \text{when } 0 \leq d \leq \frac{1}{2} \\ \frac{N-1}{2} & \text{when } \frac{1}{2} \leq d < 1 \end{cases} \]

Also we have assumed that the FIR filter coefficients are truncated from the beginning of the ideal impulse response. This is not only possible choice. The index \( M \) of the first non-zero sample should be chosen in the following way

\[ M = \begin{cases} \text{round}(D) - \frac{N}{2} & \text{for } N \text{ even} \\ \lfloor D \rfloor - \frac{N-1}{2} & \text{for } N \text{ odd} \end{cases} \]  

Where \( (\cdot) \) denotes the operation of rounding to the nearest integer and \( \lfloor . \rfloor \) is the greatest integer function. The FIR filter \( h[n] \) will be causal if \( M \geq 0 \) and non-causal if \( M \leq 0 \) and thus non-realizable and integer must be added to \( D \) in order to shift the central point of the sin C function so that the requirement of equation is fulfilled. This is equivalent to adding unit delays to the delay line.
However unit delays cannot be added if the actual delay $D$, including the integer part; is important. Another method to make the FIR filter causal is to choose a smaller value for $N$, i.e., to design a filter of lower order. Obviously this will make the approximation less accurate [8].

Another variation of the basic L.S. design is to use a reduced band width with a smooth transition band function. This will make the impulse response of the FD element delay fast. The IR will be reduced, since the discontinuity is not as sharp as originally. The good choice for the transition band is a low order spline multiplied by $e^{-j\omega D}$. Using this design the magnitude response of the FD FIR filter will remain constant with high precision.

**5. DESIGN FOR MAXIMALLY FLAT FIR AND IIR FRACTION DELAY FILTER**

We consider a linear digital filter given by the transfer function of an ideal fractional delay filter.

$$H_{\text{ideal}}(Z) = Z^{-a} a \in R$$

Having group delay

$$\tau = \frac{d}{dt}(-\angle H_{\text{ideal}}(e^{j\omega}))$$
$$= \frac{d}{d\omega}((\omega\alpha) = \alpha$$

**6. DESIGN OF FIR FILTER USING TAYLOR SERIES**

We consider the series expansion of $Z^a$ in a polynomial of order $N$

$$T(z) = \sum_{k=0}^{N-1} a_k (z - z_0^{-1})^k = \sum_{k=0}^{N-1} b_k z^{-k} = z^a$$

where

$$a_k = \frac{1}{k!} \frac{d^k}{dz^k} z^a / z = z_0^{-k}$$

and

$$b_k = \sum_{m=k}^{N} a_m \left( \frac{m}{k} \right) (-z_0)^{k-m}$$

The polynomial $T(z)$ given by the $\{b_k\}$ in the equation 16 represents non causal filter. However replacing $z$ by $z^{-1}$ we gets a causal filter [5].

$$H(z) = T(z^{-1})$$

and

$$H(z) = \sum_{k=0}^{N-1} a_k (z^{-1} - z_0^{-k})^k = \sum_{k=0}^{N-1} b_k z^{-k} = z^a$$

This is possible because for all $z$ on the unit circle the approximation error is complex conjugate to the error by approximating $z^a$ so the accuracy not get changed. Further the error $H(z) - H_{\text{ideal}}(z)$ vanishes when $z = z_0$. Then the value of $b_k$ as

$$b_k = z_0^{-\alpha} \prod_{m=k}^{N} \frac{\alpha - n}{k - n} \text{ when } n \neq k$$
In special case $z_0 = 1$, the factor $z_0^{k-a}$ in equation 20 is 1 and reduce $b_k$ for the co-efficient of maximally flat FIR F.D filter which is known as Lagrange Interpolation.

This filter is derived by setting to zero the first $N$ derivatives of $E(z) = H(z) - H_{ideal}(z)$ with respect to $\omega$ at a point $\omega = \omega_0$. Then

$$b(k) = e^{j\omega_0(k-a)} \prod_{n=0}^{N-1} \frac{\alpha - n}{k - \alpha} \quad \text{when} \quad n \neq k \quad (21)$$

7. DESIGN OF IIR FD FILTER

The general form of all pass filter can be written as

$$H(z) = \frac{z^{-N}D(z^{-1})}{D(z)} \quad (22)$$

Where $D(z)$ is a polynomial of $N$th order, which can be written as

$$D(z) = D_\zeta(z) = \sum_{k=0}^{N} a_k(\zeta)z^{-k} \quad (23)$$

Where $\zeta$ can be denoted as the point of perfect approximation [3]. The coefficient for a purely recursive FD filter are derived under maximally flat if $z = \zeta = 1$ as

$$a_k(1) = (-1)^k \binom{N}{k} \prod_{n=0}^{N} \frac{\alpha - N + n}{\alpha - n + k} \quad (24)$$

It is a purely recursive filter $H(z) = H(\zeta) = 1/D_\zeta(z)$ having group delay $(\alpha - N)/2$.

The response of this filter shown below.

**CONCLUSION**

The design of high frequency F.D. filter is difficult if a very small delay is required. It is possible to use a higher sampling rate if accurate and small fractional delays are needed. Its help the approximation task in two ways.

1. The unit delay are scaled down by new sampling interval.
2. The normalized approximation bandwidth is reduced.

Comparison between FIR and IIR Filter

It is seen that the bandwidth of FIR and IIR filter can be increased significantly by varying $z_0$ on the positive real axis, with the filter remaining real valued. However, the maximum bandwidth resulting for the FIR filter are equivalent to those resulting for IIR filter all pass filter, which suggest that given a bandwidth required by the application. In regard to the computational load FIR filter have more advantages than IIR Filter.

**REFERENCES**


