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The Fixed Points of Mobius Transformation

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Abstract: In complex analysis, a Mobius transformation of complex plane is a rational function of one complex variable z .Geometrically ,a Mobius transformation can be obtained by performing stereographic projection form plane to unit two-sphere ,rotating and moving the sphere to a new location and orientation in space and performing stereographic projection to the plane .These transformations preserves angles ,maps every staright line to a line or circle ,map every circle to a line or circle.

Keywords: Mobius transformation, Fixed points, cross ratio, translation, dilation, inversion, normal form.

1. Introduction

In this paper, I have provided a brief introduction on Mobius transformation, fixed points and some properties of this kind of transformation. Mobius transformation have applications to problems in physics, Engineering and Mathematics.

1.1 Definition: A Bilinear transformation or Mobius transformation of the plane is a map $f: C_\infty \to C_\infty$ (where C_∞ is extended complex plane i.e. $C \cup \{\infty\}$)

$w = f(z) = \frac{az+b}{cz+d}$, a,b,c,d ε C, ad-bc ≠ 0 which associates a unique point of w-plane to any point of z-plane except z =-d/c, c ≠ 0 (where z is a complex variable). In case ad-bc= 1 then this transformation is called normalized transformation.[1]

1.2 Inverse of Mobius transformation : Let $f(z) = \frac{az+b}{cz+d} = w$, ad-bc ≠ 0 be Mobius transformation then w(cz+d) = az+b ⇒ z(wc-a) = b-wd

$\Rightarrow f^{-1}(w) = z = \frac{wd-b}{-cw+a}$ is called inverse transformation which associates unique point of z-plane to any point of w-plane except w = a/c where c ≠ 0[2]

1.3 The Group of Mobius transformation :

Let $w = f(z) = \frac{az+b}{cz+d}$ be a Mobius transformation .The inverse function

$f^{-1}(w) = z = \frac{wd-b}{-cw+a}$ is again a Mobius transformation ,that is, $f \circ f^{-1} = I$, I is identity transformation. Further, we see that composition of two transformation $f_1 \circ f_2(z)$, where $f_i(z) = \frac{a_i z+b_i}{c_i z+d_i}$, i=1,2,that is,

$f_1 o f_2$ (z) = f_1 (f_2 (z)) = $\frac{a_1 f_1(z)+b_1}{c_1 f_1(z)+d_1} = \frac{a_1\frac{a_2z+b_2}{c_2z+d_2}+b_1}{c_1\frac{a_2z+b_2}{c_2z+d_2}+d_1} = \frac{a_1a_2z+a_1b_2+b_1c_2z+d_2b_1}{c_1a_2z+c_1b_2+d_1c_2z+d_1d_2} = \frac{(a_1a_2+b_1c_2)z+(a_1b_2+d_2b_1)}{(c_1a_2+d_1c_2)z+(c_1b_2+d_1d_2)}$

is again a Mobius transformation .On the other hand, foI(z) = Iof(z) = f(z).Also,($f_1 o f_2$)of_3 = f_1o(f_2of_3) So,set of all Mobius transformation is a grup with respect to the composition.[3]

1.4 Theorem: Every Mobius transformation consists of four composition functions.

Proof: let w = f(z) = $\frac{az+b}{cz+d}$,such that, ad-bc ≠ 0 be any Mobius transformation let

a.) $f_1(z) = z+\frac{d}{c}$ i.e, translation by $\frac{d}{c}$

b.) $f_2(z) = \frac{1}{f_1(z)} = \frac{1}{z+\frac{d}{c}}$, i.e. inversion and reflection with respect to real axis.

c.) $f_3(z) = \frac{-ad+bc}{c^2} f_2(z)$ i.e. dilation and rotation

d.) $f_4(z) = f_3(z)+\frac{a}{c}$ i.e translation by $\frac{a}{c}$

consider $f_4(z) = f_3(z)+\frac{a}{c} = \frac{a}{c}+\frac{-ad+bc}{c^2} f_2(z) = \frac{a}{c}+\frac{-ad+bc}{c^2}\frac{1}{f_1(z)}$

$= \frac{a}{c}+\frac{-ad+bc}{c^2}\frac{1}{z+\frac{d}{c}} = \frac{a}{c}+\frac{\frac{a}{c}[\frac{b}{a}-\frac{d}{c}]}{z+\frac{d}{c}} = \frac{az+b}{cz+d}$ = f(z) , that is, every Mobius transformation is composition of translation, dilation, inversion

2. Fixed points

2.1 Definition: The points which coincide with their transformations are called fixed points of the transformation, that is, fixed points of the transformation w = f(z) = $\frac{az+b}{cz+d}$ are obtained by solving f(z) = z i.e. w = z[4]

thus z = $\frac{az+b}{cz+d}$ ⇒ cz^2-(a-d)z-b = 0 and applying quadratic formula ,the roots are:

$z = \frac{(a-d)\pm\sqrt{(a-d)^2+4bc}}{2c}$. let $\alpha = \frac{(a-d)+\sqrt{(a-d)^2+4bc}}{2c}$, $\beta = \frac{(a-d)-\sqrt{(a-d)^2+4bc}}{2c}$

then α, β are distinct points and are fixed points for the transformation.

2.2 Normal form : Mobius transformation are also sometimes written in terms of their fixed points in so called normal form.[5]

2.3 Theorem: If there are two distinct fixed points α and β of Mobius transformation w= f(z) = $\frac{az+b}{cz+d}$ then the transformation take the form

$$\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right)$$

Proof: As seen in theorem 2.1 ,distinct fixed points α,β are given by

$\alpha = \frac{(a-d)+\sqrt{(a-d)^2+4bc}}{2c}$, $\beta = \frac{(a-d)-\sqrt{(a-d)^2+4bc}}{2c}$ since α,β are the roots of quadratic equation $cz^2-(a-d)z-b = 0$

$\Rightarrow c\alpha^2-(a-d)\alpha-b = 0$ and $c\beta^2-(a-d)\beta-b = 0$

Thus, $c\alpha^2 - a\alpha = b-d\alpha$, $c\beta^2 - a\beta = b-d\beta$

Consider, $\frac{w-\alpha}{w-\beta} = \frac{\frac{az+b}{cz+d}-\alpha}{\frac{az+b}{cz+d}-\beta} = \frac{az+b-\alpha cz-\alpha d}{az+b-\beta cz-\beta d} = \frac{(a-\alpha c)z+(b-d\alpha)}{(a-\beta c)z+(b-\beta d)}$

$= \frac{(a-\alpha c)z+c\alpha^2-a\alpha}{(a-\beta c)z+c\beta^2-a\beta}$

$= \frac{(z-\alpha)(a-\alpha c)}{(z-\beta)(a-c\beta)} = k\frac{(z-\alpha)}{(z-\beta)}$ where $k=\frac{a-\alpha c}{a-\beta c}$ is normal form of the transformation.

In case of two distinct fixed points, a transformation is hyperbolic if k >0 and elliptic if $k = e^{i\alpha}$,$\alpha \neq 0$and loxodormic if $k=ae^{i\alpha}$ where $a \neq 1,\alpha \neq 0$and a,α are both real numbers, a >0.[6]

2.4 Theorem : If there is only one finite fixed point say α of Mobius transformation w=f(z) $= \frac{az+b}{cz+d}$ then transformation takes the form

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$$

Proof: Since for fixed point, we put $w = z \Rightarrow z= \frac{az+b}{cz+d} \Rightarrow cz^2-(a-d)z-b = 0$

On solving, we get: $z = \frac{(a-d)\pm\sqrt{(a-d)^2+4bc}}{2c}$ are two fixed points but by given condition, there is only one finite fixed point $\Rightarrow (a-d)^2 +4bc = 0$

$\Rightarrow \alpha = \frac{a-d}{2c} \Rightarrow 2\alpha c = a-d \Rightarrow d = a-2\alpha c$. Again α is a root of equation

$cz^2-(a-d)z-b = 0 \Rightarrow c\alpha^2-(a-d)\alpha-b = 0 \Rightarrow c\alpha^2-a\alpha = b-d\alpha$

now, $\frac{1}{w-\alpha} = \frac{1}{\frac{az+b}{cz+d}-\alpha} = \frac{cz+d}{az+b-cz\alpha-\alpha d} = \frac{cz+b}{(a-c\alpha)z+(b-\alpha d)} = \frac{cz+a-2\alpha c}{(a-c\alpha)z+(c\alpha^2-a\alpha)} = \frac{cz+a-\alpha c-\alpha c}{(z-\alpha)(a-\alpha c)} =$ $\frac{(cz-\alpha c)+(a-\alpha c)}{(z-\alpha)(a-\alpha c)} = \frac{c}{(a-\alpha c)} + \frac{1}{(z-\alpha)}$, taking $k = \frac{c}{a-\alpha c} \Rightarrow \frac{1}{w-\alpha} = k+ \frac{1}{z-\alpha}$,which is normal form of transformation .In case of one finite fixed point, the transformation is called parabolic.[7]

3. Cross Ratio

3.1 Definition : In geometry ,the cross ratio ,also called double ratio is a number associated with a list of four collinear points ,particularly points on a projective line. If z_1,z_2,z_3,z_4 are four distinct complex numbers then the ratio $\frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)}$

is called cross ratio of z_1,z_2,z_3,z_4 and it is denoted by $(z_1, z_2; z_3, z_4)$[8]

3.2 Theorem : A Mobius transformation preserves cross ratio

Proof: Let $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be Mobius transformation .Let this transformation transforms z_1,z_2,z_3,z_4 to corresponding points w_1,w_2,w_3,w_4 in w-plane $\Rightarrow w_i = \frac{az_i+b}{cz_i+d}$, i=1,2,3,4

$$\Rightarrow w_i - w_j = \frac{az_i+b}{cz_i+d} - \frac{az_i+b}{cz_i+d} = \frac{(az_i+b)(cz_j+d)-(az_j+b)(cz_i+d)}{(cz_i+d)(cz_i+d)}$$

$$=\frac{(ad-bc)(z_i-z_j)}{(cz_i+d)(cz_j+d)}$$

Consider $(w_1,w_2;w_3,w_4) = \frac{(w_1-w_2)(w_3-w_4)}{(w_2-w_3)(w_4-w_1)}$

$$= \frac{\frac{(ad-bc)\ (z_1-z_2)\ (ad-bc)\ (z_3-z_4)}{(cz_1+d\ )(cz_2+d)(cz_3+d)\ \ (cz_4+d)}}{\frac{(ad-bc)(z_2-z_3)\quad (ad-bc)(z_4-z_1)}{(cz_2+d)(cz_3+d)(cz_4+d)\quad (cz_1+d)}}$$

$$= \frac{(z_1-z_2)(z_3-z_4)}{(z_2-z_3)(z_4-z_1)} = (z_1, z_2; z_3, z_4)$$

3.3 Theorem : A Mobius transformation carries circles into circles

Proof: By 1.4 theorem, every Mobius transformation is composition of translations, dilations and inversions. Since translations, dilations certainly map circles into circles [9],so we only have to prove the theorem for inversion i.e. $f(z) = w = \frac{1}{z}$

Given a circle centered at z_0 with radius r, that is,$|z- z_0| = r \Rightarrow |z- z_0|^2 = r^2$

$\Rightarrow (z- z_0)(\overline{z- z_0}) = r^2 \Rightarrow (z- z_0)(\bar{z}-\bar{z}_0) = r^2 \Rightarrow z\bar{z}-\bar{z}\, z_0-\bar{z}_0 z+\bar{z}_0\, z_0 = r^2$

$\Rightarrow |z|^2-z\,\bar{z}_0- z_0\bar{z}+| z_0|^2 -r^2 = 0$

Now $w = f(z) = \frac{1}{z}$ i.e. $z = \frac{1}{w}$ so we make this substitution in our equation,

we have $|\frac{1}{w}|^2- \frac{1}{w}\,\bar{z}_0- z_0\,\frac{1}{\bar{w}}+| z_0|^2-r^2 = 0$ using $|w|^2 = w\bar{w}$, we have $1- z_0\bar{w}-\bar{z}_0 w+|w|^2(|z_0|^2- r^2) = 0$

If $r = | z_0| \Rightarrow 1-\bar{z}_0 w- z_0 w = 0$, we get a straight line in terms of w[10] and

if $r \neq | z_0|$,that is,

$| z_0|^2- r^2$ is non zero we divide our equation by it ,we obtain:

$|w|^2- z_0 w/(| z_0|^2- r^2)- \bar{z}_0\bar{w}/(| z_0|^2- r^2)+1/| z_0|^2- r^2 = 0$

We define $w_0 = \bar{z}_0/(| z_0|^2- r^2)$, $s^2 = | w_0|^2-1/(| z_0|^2- r^2)$

$= | z_0|^2/(| z_0|^2- r^2)^2- (| z_0|^2- r^2)/(| z_0|^2- r^2)^2 = r^2/(| z_0|^2- r^2)^2$ Thus our equation becomes:

$|w|^2 - \bar{w}_0 w - w_0 \bar{w} + |w_0|^2 - s^2 = 0 \Rightarrow w\bar{w} - w_0\bar{w} - w\,\bar{w}_0 + w_0\,\bar{w}_0 = s^2$

$\bar{w}(w - w_0) - \bar{w}_0\,(w - w_0) = s^2 \Rightarrow (w - w_0)(\bar{w} - \bar{w}_0) = s^2 \Rightarrow |w - w_0|^2 = s^2$

this is the equation of circle in terms of w, with centre w_0 and radius s.

3.4 Theorem : Let $f(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be any Mobius transformation other than $f(z) = z$. Show that $f(z) = f^{-1}(z)$ iff $d = -a$

Proof: First let $f(z) = f^{-1}(z)$ Since $f(z) = \frac{az+b}{cz+d}$, $f^{-1}(w) = \frac{wd-b}{-cw+a} \Rightarrow f^{-1}(z) = \frac{zd-b}{-cz+a}$

Now $f(z) = f^{-1}(z) \Rightarrow \frac{az+b}{cz+d} = \frac{-b+zd}{-cz+a} \Rightarrow (az+b)(-cz+a) = (-b+zd)(cz+d)$

$\Rightarrow -acz^2 + a^2z - bcz + ba = -bcz - bd + cd\,z^2 + z\,d^2$

$\Rightarrow -ac\,z^2 + (a^2-bc)z + ab = cd\,z^2 + (d^2-bc)z - bd$ .on comparing coefficients of z^2,z,constant terms ,we get :$-ac = cd$; $a^2-bc = d^2-bc$; $ab = -bd \Rightarrow a = -d$

Conversely, if $d = -a \Rightarrow f(z) = \frac{az+b}{cz+d} = \frac{-dz+b}{cz-a} = \frac{-(zd-b)}{cz-a} = \frac{(zd-d)}{-cz+a} = f^{-1}(z)$

Thus $f(z) = f^{-1}(z)$

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