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On Prime Ideals, the Prime Radical and M-Systems

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Abstract: A group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element and satisfy certain axioms while a ring is an algebraic structure with two binary operations namely addition and multiplication. I have tried to discuss prime ideals, prime radical and m-system in this paper.

Keywords: Prime ideals, semi-prime ideals, m-system, n-system.

1. INTRODUCTION

In this paper, I have tried to explain the concept of prime ideals in an arbitrary ring, Radical of a ring and few properties of m-system. Besides these some theorems and lemma have been raised such as "If A is an ideal in ring R then B(A) coincide with intersection of all prime ideals in R which contain A". Also some theorems and lemmas based on m-system and n-system have been established.

1.1 Prime Integer: An integer p is said to be prime integer if it has following property that if a and b are integers such that ab is divisible by p then a is divisible by p or b is divisible by p.

1.2 Prime Ideal [1]: An ideal P in ring R is said to be a prime ideal if and only if it has the following property:

If A and B are ideals in R such that $AB \subseteq P$, then $A \subseteq P$ or $B \subseteq P$

1.3 m-system: A set M of elements of a ring R is said to be an m-system if and only if it has following property:

If $a, b \in M$, these exists $x \in R$ such that $axb \in M$

1.4 Theorem: If P is an ideal in ring R, then following conditions are equivalent:

(i) P is a prime ideal

(ii) If $a, b \in R$ such that $aRb \subseteq P$ then $a \in P$ or $b \in P$

(iii) If (a) and (b) are principal ideals in R such that $(a)(b) \subseteq P$ then $a \in P$ or $b \in P$.

(iv) If U and V are right ideals in R such that $UV \subseteq P$ then $U \subseteq P$ or $V \subseteq P$

(v) If U and V are left ideals in R such that $UV \subseteq P$ then $U \subseteq P$ or $V \subseteq P$

Proof:- We first prove (i) implies (ii). Let us assume that P is prime ideal & suppose that $aRb \subseteq P$

$\Rightarrow RaRbR \subseteq P$ Now, $(RaR)(RbR) \subseteq RaR\, bR \subseteq P$, now RaR, RbR are ideals and P is prime ideal $\Rightarrow$ either $RaR \subseteq P$ or $RbR \subseteq P$

Suppose that $RaR \subseteq P$ and let $A = (a)$ Thus, $A^3 \subseteq RaR \subseteq P$ [2]

Again P is prime ideal thus we have $A \subseteq P \Rightarrow a \in P$

Similarly if $RbR \subseteq P$, it follows that $b \in P$ Thus (ii) is established.

Net we prove (ii) implies (iii) Let $(a)(b) \subseteq P$

Now, aRb $\subseteq$ (a) (b) $\subseteq$ P and (ii) implies that a ε P or b ε P $\Rightarrow$ (iii) is established.

Next we show that (iii) implies (iv): Suppose, U and V are right ideals in R such that UV $\subseteq$ P Let us assume that U $\not\subset$ P and prove that V $\subseteq$ P

Let u ε U with u $\notin$ P and v is an arbitrary element of V

Since (u)(v) $\subseteq$ UV+RUV $\subseteq$ P $\Rightarrow$ (u)(v) $\subseteq$ P

Since u $\notin$ P and (iii) implies v ε P $\Rightarrow$ V $\subseteq$ P. In a similar manner, (iii) implies (v)

It is trivial that either (iv) or (v) implies (i). Thus Proof of theorem is therefore complete.

Remarks: (1) If P is ideal in R, let us denote by C(P) the complement of P in R, that is, C(P) is the set of elements of R which are not elements of P.

(2) From above theorem (i) and (ii) asserts that P is prime ideal in R if and only if C(P) is an m-system.

(3) If R is itself a prime ideal in R then C(R) = ϕ

1.5 Prime radical [3]: The prime radical B(A) of ideal A in ring R is the set consisting of those elements r of R with the property that every m-system in R which contains r meets A (that is, has nonempty intersection with A)

Remarks (1) Obviously B(A) is ideal in R. Also A $\subseteq$ B(A)

(2) Suppose P is a prime ideal in R such that A $\subseteq$ P and let r ε B(A). If r $\notin$ P, C(P) would be an m-system containing r and therefore C(P) $\cap$ A$\neq\phi$. However, since A $\subseteq$ P, C(P) $\cap$ A= ϕ and this contradiction shows that r ε P Thus, B(A) $\subseteq$ P.

(3) A set of elements of a ring which is closed under multiplication is often called multiplicative system.

(4) If r ε R then the set { r^i / i=1,2,3,……} is a multiplicative system and hence also an m-system.[4]

1.6 Theorem: If A is an ideal in the ring R, then B(A) coincides with the intersection of all prime ideals in R which contain A.

Proof: By the above remark, B(A) is contained in every prime ideal which contains A. We will prove it by showing if r $\notin$ B(A), then there exists a prime ideal P in R such that r $\notin$ P and A $\subseteq$ P Since r $\notin$ B(A), by definition of B(A), there exists an m-system M such that r ε M and M $\cap$ A=ϕ. Now, consider set τ of all ideals K in R such that A $\subseteq$ K and M $\cap$ K = ϕ, that is,τ = { K is ideal in R / A $\subseteq$ K, M $\cap$ K = ϕ}. Now, A is ideal in R such that A $\subseteq$ A, M $\cap$ A = ϕ thus, A ετ and therefore set τ is non-empty.

By applying, zorn's Lemma to this set, there exists a maximal ideal say P , in the set.

Now, A $\subseteq$ P and M $\cap$ P = ϕ

Since r ε M and M $\cap$ P = ϕ implies r $\notin$ P

Now, the proof is complete by showing that P is prime ideal.

Let us suppose that a $\notin$ P and b $\notin$ P. Now, To prove P is prime ideal, it is sufficient to prove, (a) (b) $\not\subset$ P since P $\subseteq$ P+(a), P $\subseteq$ P+(b). Since P is maximal element of τ, thus, P+(a) and P+(b) do not belong to τ thus, (P+(a)) $\cap$ M $\neq\phi$ and (P+(b)) $\cap$ M $\neq\phi$ Thus there exists m_1 of M such that m_1 ε P +(a) and there exists $_{m2}$ of M such that m_2 ε P+(b).

Since M is an m-system, there exists an element x of R such that m_1xm_2ε M.Also m_1xm_2ε (P+(a))(P+(b)) now if (a)(b) $\subseteq$ P therefore (P+(a))(P+(b)) $\subseteq$ P $\Rightarrow$ m_1xm_2 ε P Also m_1xm_2ε M implies m_1xm_2ε M $\cap$ P and we have M $\cap$ P $\neq\phi$ which is a contradiction as M $\cap$ P = ϕ

Thus, (a)(b) $\not\subset$ P and therefore P is prime ideal.

1.7 Semi-prime ideal [5]: An ideal Q in a ring R is said to be a semi-prime ideal if and only if it has following property: If A is an ideal in R such that $A^2 \subseteq Q$, then $A \subseteq Q$.

1.8 n-system: A set N of elements of a ring R is said to be an n-system if and only if it has following property: If $a \varepsilon N$, there exists $x \varepsilon R$ such that $axa \varepsilon N$.

Remarks: (1) Every Prime ideal is semi-prime ideal.

(2) Clearly m-system is also an n-system.

(3) An ideal Q is semi-prime ideal if and only if C(Q) is an n-system.[6]

II. THE PRIME RADICAL OF A RING

The prime radical of the zero ideal in ring R may be called the prime radical of ring R and prime radical of ring R is defined as $B(R) = \{r / r \varepsilon R$, every m-system in R which contains r also contains o$\}$[7]

2.1 Prime ring: A ring R is said to be a prime ring if and only if the zero ideal is prime ideal in R. That is, ring R is prime ring iff either of following conditions holds: If A and B are ideals in R such that AB = (0) then A = (0) or B = (0)

Remark: (1) If R is commutative ring then R is a prime ring iff it has no non-zero divisors of zero.

2.2 Lemma: If $r \varepsilon B(A)$ then there exists a positive integer n such that $r^n \varepsilon A$

III SEMI-PRIMAL AND COMMUTATIVE IDEALS

If A is an ideal in commutative ring R then $B(A) = \{r \mid r^n \varepsilon A$ for some positive integer n$\}$

3.1 Theorem: An ideal Q in ring R is semi-prime ideal in R if and only if residue class R/Q contains no non zero nilpotent ideals.

Proof: Let θ be natural homomorphism of R onto R/Q with Kernel Q. Suppose that Q is Semi-prime ideal in R and U is nilpotent ideal in R/Q, say $U^n = (0)$ then

$U^n\theta^{-1} = \{a / \theta(a) \varepsilon U^n=(0)\}$ = Kernel of $\theta = Q$

And it follows that $(U\theta^{-1})^n \subseteq U^n\theta^{-1} = Q$

Since Q is semi-prime ideal, therefore $U\theta^{-1} \subseteq Q$ [8]

Hence U=(0). Thus R/Q contains no non-zero nilpotent ideal. Conversely, suppose that R/Q has no non-zero nilpotent ideals and A is ideal in R such that $A^2 \subseteq Q$ then $(A\theta)^2 = A^2Q = (0)$ hence $A\theta = 0 \Rightarrow A \subseteq Q \Rightarrow Q$ is semi-prime ideal.

3.2 Theorem: If Q is an ideal in ring R, all of following conditions are equivalent:

(i) Q is a semi-prime ideal

(ii) If $a \varepsilon R$ such that $aRa \subseteq Q$ then $a \varepsilon Q$

(iii) If (a) is principal ideal in R such that $(a)^2 \subseteq Q$ then $a \varepsilon Q$

(iv) If U is a right ideal in R such that $U^2 \subseteq Q$ then $U \subseteq Q$

(v) If U is a left ideal in R such that $U^2 \subseteq Q$ then $U \subseteq Q$

3.3 Theorem: If N is an n-system in the ring R and $a \varepsilon N$, there exists an m-system M in R such that $a \varepsilon M$ and $M \subseteq N$

Proof: Let $M = \{a_1, a_2, a_3, \ldots\ldots.\}$ where the elements of this sequence are defined inductively as follows:

First, we define $a_1 = a$, since $a \varepsilon N \Rightarrow a_1 \varepsilon N$

Now, N is an n-system therefore, $a_1 R a_1 \cap N \neq \phi$

Again $a_2 \varepsilon$ N, N is an-system $\Rightarrow a_2$ R $a_2 \cap N \neq \phi$

In general, if a_i has been defined, with $a_i \varepsilon$ N, we choose a_{i+1} as element of a_i R $a_i \cap N$. Thus a set M is defined such that a ε M and M $\subseteq$ N. To complete the proof, we need to show that M is an m-system.

Suppose a_i, $a_j \varepsilon$ M, For our convenience, let us assume that i < j then $a_{j+1} \varepsilon$ a_j $Ra_j \subseteq$ $a_i Ra_j$ and $a_{j+1} \varepsilon$ M implies a_i R $a_j \subseteq$ M. A similar argument takes in case of i > j, So we conclude that M is an m-system and this completes the proof.

IV SEMI PRIME IDEALS

4.1 Theorem: An ideal Q in ring R is semi-prime ideal in R iff B(Q)=Q

Proof: Let Q be semi-prime ideal clearly, Q $\subseteq$ B(Q) Let us assume that Q $\subset$ B(Q) and seek a contradiction .Since Q $\subset$ B(Q), So let a ε B(Q) with a $\notin$ Q

Now, Q is semi-prime ideal therefore C(Q) is an n-system and a $\notin$ Q $\Rightarrow$ a ε C(Q) therefore above theorem there exists an m-system M such that a ε M $\subseteq$ C(Q)

Now, a ε B (Q) $\Rightarrow$ by definition of B(Q) , M $\cap$ Q $\neq \phi$

Since M $\subseteq$ C(Q) $\Rightarrow$ M $\cap$ Q $\subseteq$ C(Q) $\cap$ Q = ϕ

$\Rightarrow$ M $\cap$ Q = ϕ, which give contradiction and this gives proof of the theorem.

4.2 Theorem: If the descending chain condition (d.c.c) for right ideals holds in a ring R, every nil right ideal in R is nilpotent.

Proof: Let N be a non-zero nil right ideal in R

Since $N \supseteq N^2 \supseteq N^3 \supseteq$ …………………,

by descending chain condition, there exists a positive integer n such that $N^n = N^{n+1} = N^{n+2} =$ …..

We shall prove that N^n = (0) .for our convenience, let us set $M = N^n$. Let us assume that M $\neq$ (0). Since M^2 = MM = $N^n N^n = N^{2n} = N^n$ = M $\Rightarrow$ M^2 = MM = M$\neq$ (0)

Now, consider the set τ of all right ideals A in R such that AM $\neq$ (0), that is,

τ = { A is right ideal in R / AM $\neq$(0)}

Since N is right ideal in R and MM $\neq$(0)

$\Rightarrow$ M $\varepsilon\tau$ implies that τ is non-empty set

Now, d.c.c. assume that there exists a right ideal B which is minimal ideal of τ [9]

Since B $\varepsilon\tau \Rightarrow$ BM $\neq$ (0), there exists an element b of B s.t bM $\neq$ (0).

Now, bM is right ideal in R such that bM $\subseteq$ B and (bM)M = bMM = bM^2 = bM $\neq$ 0

$\Rightarrow$ bM is an element of set τ, bM $\subseteq$ B . Now, B is minimal element of τ, we have bM = B.

In particular, these exists an element m of M s.t bm = b

Now, bm^2 = bmm = (bm)m = bm = b

bm^3 = b m^2m = bmm = b m^2 = bm

b = bm = b m^2 = b m^3 = ………………=bm^k =bm^{k+1}=……….

Since m ε M , M is nil ideal $\Rightarrow$ m is nilpotent $\Rightarrow$ m^k = 0 for some positive integer k [10]

$\Rightarrow$ b m^k = 0 since b m^k = b

$\Rightarrow$ b = 0

Hence bM = (0) which is a contradiction and therefore, we conclude that M = (0), that is, Nn = (0) and thus, N is nilpotent.

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