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Comparison Between Varoius Methods for Solving First Order Differential Equations

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Abstract

Various numerical methods are used in solving first order differential equations. This paper mainly presents three methods namely Euler's, Taylor's and First to higher order Runge-Kutta Methods.Also comparison between the above mentioned methods have been discussed. Apart from these local truncation error and global errors are also explained.

1 Introduction

Differential equation [4] is an equation involving independent variables, dependent variables and derivatives of dependent variables w.r.t. independent variables. If there is single independent variable then differential equation is called ordinary differential equation, otherwise it is called partial differential equation. There are two types of ordinary differential equation : first is initial value problem and another is boundary value problem.

In our world, things change, and describing how they change often ends up as a differential equation. Differential equations can describe how populations change, how heat moves, how spring vibrates, how radioactive material decays and much more. They are a very natural way to describe many things in the universe. Differential equations are one of the most important mathematical tools used in making models in physical sciences, engineering. Many problems of modern world can be formulated in terms of differential equations. Many authors [1, 2, 3, 5] have attempted to solve initial value problem to obtain high accuracy in solution by using numerical methods such as Euler' method, Taylor's method and Runge-Kutta methods etc. Taylor's method is easy in understanding but not very practical. The accuracy of this method falls as we proceed for more complicated functions. On the other hand, Runge-Kutta method is more reliable as it gives reliable starting values and most stable when there are functions having complicated higher derivatives. In this paper, numerical methods are explained for solving first order ordinary differential equation with initial value problem. The numerical methods for first order differential equation can be extended to a system of first order differential equations. Also higher order differential equations can be solved using the concept that nth order differential equation is equivalent to system of differential equations of first order.

2 Euler's Method

Euler's method is a numerical method to solve first order, first degree differential equation with a given initial value problem. It is the most basic explicit method for numerical integration of ordinary differential equtions and it is the simplest Runge-Kutta method. It is not an efficient numerical method but many of the ideas involved in the numerical solution of the differential equations are introduced most simply with it. The differential equation

$$x'(t) = g(t, x(t)), t_0 \leq t \leq b \textit{ with the initial condition } x(t_0) = x_0 \quad (1)$$

is called initial value problem. Above differential equation is first order differential equation which can be linear or non-linear. It can represent a system of differential equations also if $g(t), x(t)$ are vectors. In most practical applications of ordinary differential equation independent variable 't' represents time variable with t_0 as initial time and $x(t)$ denoting the true solution of initial value problem with initial value x_0.

Numerical methods for solving above initial value problem will help to find an approximate solution $x(t)$ at a discrete set of nodes, $t_0 < t_1 < t_2 < ... < t_N \leq b$.

These nodes are taken equally spaced,that is, $t_n = t_0 + nh$, $n = 0, 1, 2, ..., N$.

Euler's method assumes the solution is written in the form of Taylor's series

$$x(t+h) \backsim x(t) + hx'(t) + \frac{h^2}{2!}x''(t) + \frac{h^3}{3!}x'''(t) + ...$$

Above expression provides good approximation if we take plenty of terms and if value of h is reasonably small.
For Euler's method, only first two terms are taken, that is,

$$x(t+h) \backsim x(t) + hx'(t)$$

or we can write $x(t+h) \backsim x(t) + hg(t, x(t))$

3 Taylor's Method

To solve initial value problem $x'(t) = g(t, x(t))$, $t_0 \leq t \leq b$, $x(t_0) = x_0$, select a Taylor approximation of certain order for order p,

$$x(t_{n+1}) \backsim x(t_n) + hx'(t_n) + ... + \frac{h^p}{p!}x^{(p)}(t_n)$$

where truncation error is

$$T_{n+1}(x) = \frac{h^{p+1}}{(p+1)!}x^{(p+1)}(\xi_n), \quad t_n \leq \xi_n \leq t_{n+1}$$

Now $x''(t) = f_t + f_x f$ and $x''(t) = f_{tt} + 2f_{tx}f + f_{xx}f^2 + f_x(f_t + f_x f)$. The formulas for higher derivatives rapidly become very complicated as differentiation order is increased. This method is tedious and time consuming.

4 Runge-Kutta Method

For solving differential equation of the form $\frac{dx}{dt} = g(t, x)$, method that is used to solve it is of the form new value = old value + slope *step size, i.e.,

$$x_{i+1} = x_i + h\phi \tag{2}$$

According to this equation, the slope estimate of ϕ is used to extrapolate from an old value x_i to a new value x_{i+1} over a distance h. Runge-Kutta method achieve the accuracy of Taylor series approach without requiring the calculation of higher derivatives. Many variations exist but all can be cast in the generalized form of equation (2) :

$$x_{i+1} = x_i + h\ \phi(t_i, x_i, h) \tag{3}$$

where $\phi(t_i, x_i, h)$ is called an increment function, which can be interpreted as a representative slope over the interval. The increment function can be written in general form as :

$$\phi = a_1k_1 + a_2k_2 + ... + a_nk_n$$

where $a's$ are constants and $k's$ are :
$k_1 = g(t_i, x_i)$, $k_2 = g(t_i + p_1h, x_i + q_{11}k_1h)$,
$k_3 = g(t_i + p_2h, x_i + q_{21}k_1h + q_{22}k_2h), ...$
$k_n = g(t_i + p_{n-1}h, x_i + q_{n-1,1}k_1h + q_{n-2,2}k_2h + ... + q_{n-1,n-1}k_{n-1}h)$
where $p's$ and $q's$ are constants. Notice that $k's$ are recurrence relationships, that is, k_1 appears in the equation for k_2, which appears in the equation for k_3, and so forth.
Various types of Runge-Kutta methods can be devised by employing different numbers of terms in the increment function as specified by n. Note that the first order Runge-Kutta method with $n = 1$ is, in fact, Euler's method. Second order Runge-kutta method use an increment function with two terms (n=2). These second order Runge-Kutta method will be exact if the solution to the differential solution is quadratic.

5 Second Order Runge-Kutta Method

The second order version of (3) is :

$$x_{i+1} = x_i + h(a_1k_1 + a_2k_2) \tag{4}$$

$$where\ \ k_1 = g(t_i, x_i) \tag{5}$$

$$k_2 = g(t_i + p_1h, x_i + q_{11}k_1h) \tag{6}$$

To determine values of a_1, a_2, p_1 and q_{11}, the second order Taylor series for x_{i+1} in terms of x_i and $f(t_i, x_i)$ is written as :

$$x_{i+1} = x_i + hg(t_i, x_i) + \frac{h^2}{2!}g'(t_i, x_i) + ... \tag{7}$$

where $g'(t_i, x_i)$ must be determined by the chain rule differentiation

$$g'(t_i, x_i) = \frac{\partial g(t,x)}{\partial t} + \frac{\partial g(t,x)}{\partial t}\frac{dx}{dt} \tag{8}$$

Using (6) and (7) , we have

$$x_{i+1} = x_i + hg(t_i, x_i) + \frac{h^2}{2!}\Big(\frac{\partial g(t,x)}{\partial t} + \frac{\partial g(t,x)}{\partial x}\frac{dx}{dt}\Big) \tag{9}$$

now, using Taylor series for a two variable function, that is, $g(t+r, x+s) = g(t,x) + r\frac{\partial g}{\partial t} + s\frac{\partial g}{\partial x} + ...$

$$g(t_i + p_1 h, x_i + q_{11}k_1 h) = g(t_i, x_i) + p_1 h\frac{\partial g}{\partial t} + q_{11}k_1 h\frac{\partial g}{\partial x} + O(h^2) \tag{10}$$

Using (8), (5) in (4) :
$x_{i+1} = x_i + ha_1 g(t_i, x_i) + a_2 hg(t_i, x_i) + a_2 p_1 h^2\frac{\partial g}{\partial t} + a_2 q_{11}h^2 g(t_i, x_i)\frac{\partial g}{\partial x} + O(h^3)$

$$= x_i + h[a_1 g(t_i, x_i) + a_2 g(t_i, x_i)] + h^2[a_2 p_1\frac{\partial g}{\partial t} + a_2 q_{11} g(t_i, x_i)\frac{\partial g}{\partial t}] + O(h^3) \tag{11}$$

Comparing like terms in equation (9) and (11) :
$a_1 + a_2 = 1$, $p_1 = \frac{1}{2}$, $a_2 q_{11} = \frac{1}{2}$
Thus, $a_1 = 1 - a_2$, $p_1 = q_{11} = \frac{1}{2}a_2$
Because there can be infinite number of values for a_2, there are infinite number of second order Runge-Kutta Methods.
Every version would yield exactly the same results if the solution to ordinary differential equations were quadratic, linear or constant.
(i) Heun method with a single corrector : If a_2 is assumed to be $\frac{1}{2}$, therefore $a_1 = \frac{1}{2}$, $p_1 = q_{11} = 1$
$x_{i+1} = x_i + \frac{h}{2}(k_1 + k_2)$
where $k_1 = g(t_i, x_i)$, $k_2 = g(t_i + h, x_i + k_1 h)$
Note that k_1 is the slope at the beginning of the interval and k_2 is the slope at the end of the interval.
(ii) Midpoint method : If a_2 is assumed to be 1 then $a_1 = 0$, $p_1 = q_{11} = \frac{1}{2}$
thus, $x_{i+1} = x_i + k_2 h$
where $k_2 = g(t_i + \frac{h}{2}, x_i + k_1\frac{h}{2})$
This is midpoint method.
(iii)Ralston's method : If a_2 is assumed to be $\frac{2}{3}$ then $a_1 = \frac{1}{3}$, $p_1 = q_{11} = \frac{3}{4}$
thus, $x_{i+1} = x_i + \frac{h}{3}(k_1 + 2k_2)$
where $k_1 = g(t_i, x_i)$, $k_2 = g(t_i + \frac{3}{4}h, x_i + \frac{3}{4}k_1 h)$

6 Third order Runge-Kutta Method

For $n = 3$, a derivation similar to that of second order Runge-Kutta method can be performed. The result of this derivation is six equations with eight unknowns. Thus the values of two of the unknowns must be specified priorily in order to determine the remaining parameters.
One common version that results is :
$x_{i+1} = x_i + \frac{h}{6}(k_1 + 4k_2 + k_3)$
where $k_1 = g(t_i, x_i)$, $k_2 = g(t_i + \frac{h}{2}, x_i + k_1\frac{h}{2})$, $k_3 = g(t_i + h, x_i - k_1 h + 2k_2 h)$
Note that if the derivative is a function of x only then this thrid order method reduces to Simpson's 1/3 rule.

7 Forth order Runge-Kutta method

The most popular Runge-Kutta are fourth order. As with the second order approaches, there are an infinite number of versions. The fourth order Runge-Kutta methods is :
$x_{i+1} = x_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
where $k_1 = g(t_i, x_i)$, $k_2 = g(t_i + \frac{h}{2}, x_i + k_1\frac{h}{2})$, $k_3 = g(t_i + \frac{h}{2}, x_i + k_2\frac{h}{2}), k_4 = g(t_i + h, x_i + k_3 h)$

8 Higher order Runge-Kutta Method

When more accurate results are required, Butcher's fifth order Runge-Kutta method is recommended.
$x_{i+1} = x_i + \frac{h}{90}(7k_1 + 32k_3 + 12k_4 + 32k_5 + 7k_6)$
where $k_1 = g(t_i, x_i)$, $k_2 = g(t_i + \frac{h}{4}, x_i + k_1\frac{h}{4})$, $k_3 = g(t_i + \frac{h}{4}, x_i + k_1\frac{h}{8} + k_2\frac{h}{8})$, $k_4 = g(t_i + \frac{h}{2}, x_i - k_2\frac{h}{2} + k_3 h)$, $k_5 = g(t_i + \frac{3}{4}h, x_i + k_1\frac{3h}{16} + k_4\frac{9h}{16})$, $k_6 = g(t_i + h, x_i - k_1\frac{3h}{7} + k_2\frac{2h}{7} + k_3\frac{12h}{7} - k_4\frac{12h}{7} + k_5\frac{8h}{7})$

9 Error

Approximation of a function allows possibility of deviation from correct value of function. Error represents amount by which an approximation differs from an exact solution.
Local truncation error : Local truncation error of a numerical method is an estimate of the error introduced in a single iteration of the method.
If $x_1, x_2, ..., x_N$ are numerically computed values and $x(t_1), x(t_2), ..., x(t_N)$ refers to corresponding exact values then local truncation error $= x(t_{n+1}) - x_{n+1}$
It represents the terms neglected by truncating Taylor's series but it is not the error that we get from method.
Global Error : Global error of a numerical method is an estimate of the error involved in the whole process.
$E_n = |x(t_n) - x_n|$

Example : Consider the initial value problem $\frac{dx}{dt} = -x$, $x(0) = 1$ on the interval $0 \leq t \leq 1$. The exact solution of the given problem is given by $x(t) = e^{-t}$. The local truncation error and global error for Euler's method, Taylor's method and Higher order Runge-kutta method are shown in the following table.

t_i	Euler's Method		Taylor's Method		Higher order RK method		Exact soln
	Approx.	Local error	Approx.	Local error	Approx.	Local error	
0.1	.9	.004837416	.905	-.00016258	.904837414	2.0e-9	.904837416
0.2	.81	.008730752	.819025	-.00028924	.818730753	-9.9999e-10	.818730752
0.3	.729	.011818219	.741217625	-.00039941	.740818013	2.06e-7	.740818219
0.4	.6561	.014220045	.6708	-.00047995	.670319876	1.69e-7	.670320045
0.5	.59049	.01604066	.6071	-.00056934	.606530496	1.64e-7	.60653066
0.6	.531441	.01737	.5494	-.00058836	.548811488	1.48e-7	.548811636
0.7	.4782969	.018288403	.4972	-.00061469	.496347352	.000237951	.496585303
0.8	.4304671	.018861864	.44997	-.00064104	.449113657	.000215307	.449328964
0.9	.387420489	.019149171	.4072	-.00063034	.406375952	.000193708	.40656966
1.0	.34867844	.0192006	.368516	-.000637	.367704167	.000175274	.367879441

10 Conclusion

In this paper, Euler's method, Taylor's method, Higher order Runge-kutta methods are used for solving first order ordinary differential equation with initial value problem. The numerical solution obtained for the above example are in good agreement with solution obtained form higher order RK method. Euler's method and Taylor's method were found to be less accurate with the numerical results that were obtained from approximation solution to the exact solution. Therefore it may be concluded that the higher order Runge-Kutta method is most powerful and efficient in finding the numerical solution in comparison to exact solution.

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