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A PAIR OF HIGHER ORDER SYMMETRIC NONDIFFERENTIABLE MULTIOBJECTIVE MINI-MAXMIXED PROGRAMMING PROBLEMS

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ABSTRACT

A pair of higher order symmetric non differentiable minimax mixed programming problem where each objective function contains support function of compact convex set in R^n, is formulated. Under higher order F-convexity assumption, weak, strong and converse symmetric duality theorems related to a properly efficient solution and self-duality are proved

Keywords: efficient solution, self duality, Fconvexity, support function, minimax-mixed, Higher-order symmetric

1. INTRODUCTION

Symmetric duality in nonlinear programming problem was first introduced by Dorn [1] who defined a mathematical programming problem and it's dual to be symmetric if the dual is the primal problems. Later Dantzing, Eisinberg and Cottle [2] and Mond [3] formulated a pair of symmetric dual programs for scalar function $f(x, y)$ that is convex in the first variable and that is concave in the second variable respectively. Balas[4] generalized duality for linear and nonlinear mixed integer programming problems.

Under the weaker convexity assumptions imposed on f, Mond and Weir [5] formulated a pair of minimax symmetric dual programs. Chandra and Kumar formulated a pair of symmetric dual minimax integer Programs, in which some primal and dual variables are constrained to belong to the set of integers for arbitrary cones. Kim and Song [6] also, formulated two pair of nonlinear multiobjective mixed integer programs for arbitrary cones and established the duality theorems. Mond [7] first formulated second order symmetric dual models, introduced the concept of second order convex function and proved second order symmetric duality theorems. Bector and Chandra [8] established the second order pseudo convexity and pseudo- concavity assumptions. Devi [9] formulated a pair of second order symmetric dual programs and established duality results involving second order invex functions. Pandey [10] introduced second order n-invex function for multiobjective fractional programming problem and established weak and strong duality theorems. Mond and Schecter [11] constructed two new symmetric dual pairs in which the objective function contain a support functions of compact convex set in R^n and are therefore nondifferentiable. Under the second order F-pseudo convexity assumptions, Hou and Yang [12] gave the second order symmetric duality.

Higher order duality in nonlinear programs have been studied by some researchers. Mangasarian [13] formulated a class of higher order dual problems for nonlinear programming problem. Mond and Zhang obtained duality results for various higher order dual programming problems under higher order invexity assumption, such as higher order type-1, higher order pseudo type-1, and higher order quasi type -1 conditions. Mishra and Rueda [14] gave various duality results which included Mangasarian higher order duality and Mond-Weir higher order duality. Chen [15] also discussed the duality theorems under the higher order F-convexity for a pair of nondifferentiable programs. Chen first gave a pair of nondifferentiable multiobjective functions contains a support function of compact convex set in R^n and discussed the symmetric duality for multiobjective minimax mixed integer programming problems.

2. PRESENT WORK

In this chapter, a pair of higher order symmetric nondifferentiable mltiobjective mini-max mixed programming problems by introducing a differentiable function is formulated, where each of objective functions contains a support function of a compact convex set in R^n. For a differentiable function $h : R^n \times R^n \to R$ the definitions of the higher order F-convexity (F-pseudo convexity, F-pseudo convexity) with respect to h are introduced. Then all known other generalized invexity, such that, type-1 invexity and higher order type-1 invexity can be put into the category of the higher order type-1 invexity can be put into the category of the higher order F-invex functions by taking certain appropriate transformations of F and h. Under these the higher-order

F-convexity assumption, the higher order weak , higher order strong and higher converse symmetric duality theorems related to a properly efficient solution and self duality are proved.

3. NOTATION AND DEFINITION

Throughout this chapter R^n and R^m are n-dimensional and m-dimensional Euclidian spaces respectively. ${R^n}_+$ and ${R_+}^m$ are non negative orthants of R^n and R^m respectively. Let U and V be two arbitrary sets of integers in $R^{n_1} (0 \le n_1 \le n)$ and $R^{m_1} (0 \le m_1 \le m)$ respectively and C_1 and C_2 are closed convex cones in R^{n_2} and R^{m_2}. Let $x \in R^n$ and $y \in R^m$. Without loss of generality, suppose the first n_1 components of x and the first m_1 components of y are constrained to be integersand write $(x, y) = (x^1, x^2, y^1, y^2)$ where $x^1 \in U$ and $y^1 \in V$, $x^2 \in C_1$ and $y^2 \in C_2$. where $n = n_1 + n_2$ and $m = m_1 + m_2$. For a real-valued twice differentiable function $g(x, y)$ defined on an open set $R^n \times R^m$, denote by $\nabla_{x^2} g(\bar{x}, \bar{y})$ the gradient vector of g with respect to x^2 at $(\bar{x}, \bar{y})$, $\nabla_{x^2 x^2} g(\bar{x}, \bar{y})$, the Hessian matrix with respect to x^2 at $(\bar{x}, \bar{y})$, similarly $\nabla_{x^2 y^2} g(\bar{x}, \bar{y})$ and $\nabla_{y^2 y^2} g(\bar{x}, \bar{y})$ are also defined.

Let C be a compact convex set in R^n. The support function of C is defined by

$$s(x \mid C) = \max\{x^T y \mid y \in C\}$$

A support function, being convex and everywhere finite, has a sub differential, that is there exists $z \in R^n$ such that $s(x \mid C) \ge s(x \mid C) + z^T (y - x) \mid C)$ for all $y \in C$.

The sub differential of $s(x \mid C)$ is given by $\partial s(x \mid c) = \{z \in C \mid z^T x = s(x \mid C)\}$.

For any set $D \subset R^n$, the normal cone to D at a point $x \in D$ and is defined as

$$N_D(x) = \{y \in R^n \mid y^T (z - x) \le 0, \forall z \in D\}.$$

It is obvious that for a compact convex set C, $y \in N_C(x)$ iff $s(x \mid C) = x^T y$, or equivalently, $x \in \partial s(y \mid C)$

Consider the following multiobjective programming problem

(MOP) Minimize $f(x)$

subject to $g(x) \le 0$, $x \in X$, where $f : R^n \to R^k$, $g : R^n \to R^t$ and $X \subset R^n$.

We denote the set of feasible solutions of (MOP) by $P = \{x \in X \mid g(x) \le 0\}$.

Definition 3.1: A point $\bar{x} \in P$ is said to be an efficient solution of (MOP) if there exists no other $x \in P$ such that $f(\bar{x}) - f(x) \in R^k \setminus \{0\}$, that is $f_i(x) \le f_i(\bar{x})$ for all $i \in \{1, 2, 3, \dots k\}$, and at least one $j \in \{1,2,3,\dots k\}$, $f_j(x) \le f_j(\bar{x})$; $\bar{x} \in P$ is said to be a weak efficient solution of (P) if there exists no other $x \in P$ such that for all $i \in \{1,2,3,\dots k\}$, $f_i(\bar{x}) > f_i(x)$.

Definition 3.2: $\bar{x} \in P$ is said to be a Geoffrion properly efficient solution of (P), if $\bar{x}$ is an efficient solution, and there exists a real number $M > 0$ such that for all $i \in \{1,2,3,\dots.p\}$, $x \in P$ and $f_j(x) < f_j(\bar{x})$, then $f_j(\bar{x}) - f_j(x) \le M[f_j(x) - f_j(\bar{x})]$ for some $j \in \{1,2,3,\dots k\}$ such that $f_j(x) < f_j(\bar{x})$.

Lemma 3.1: If $x \in P$ is a properly efficient solution of (MOP), there exist $a = (a_1, a_2, \dots a_n)^T \in R^n$ and $\beta = (\beta_1, \beta_2, \dots \beta_m)^T \in R^m$ such that

$$\sum_{i=1}^{n} a_i \nabla_x f_i(\bar{x}) + \sum_{j=1}^{m} \beta_j \nabla_x g_j(\bar{x}) = 0,\ a \ge 0, \beta \ge 0,\quad (a^T, \beta^T) \ne 0.$$

Definition 3.3: A function $F : X \times X \times R^n \to R$ (where $X \subseteq R^n$) is sublinear with respect to the third variable if for all $(x, u) \in X \times X$

(*i*) $F(x, u; a_1 + a_2) \le F(x, u; a_1) + F(x, u; a_2)$, for all $a_1, a_2 \in R^n$.

(*ii*) $F(x, u; \alpha a) = \alpha F(x, u; a)$, $a \ge 0$, $\alpha \ge 0$, for all $a \in R^n$.

Definition 3.4: Suppose that $h : X \times R^n \to R$ is a differentiable function, F is sub linear with respect to the third argument. We say that

(*i*) f is said to be higher order F-convexity in $u \in X$ with respect to h, if for all $(x, p) \in X \times R^n \Rightarrow f(x) - f(u) \ge F(x, u; \nabla_x f(u) + \nabla_p h(u, p)) - p^T \{\nabla_p h(u, p)\}$

(*ii*) f is said to be higher order F-pseudo convexity in $u \in X$ with respect to h, if, for all $(x, p) \in X \times R^n$. we have

$$F(x, u; \nabla_x f(u) + \nabla_p h(u, p)) \ge 0$$

$$\Rightarrow \qquad f(x) - f(u) \ge h(u, p) - p^T \{\nabla_p h(u, p)\}$$

(*iii*) f is said to be higher -order F-quasi-convex in $u \in X$ with respect to h, if for all $(x, p) \in X \times R^n$, We have

$$f(x) \le f(u) + h(u, p) - p^T \{\nabla_p h(u, p)\}$$

$$\Rightarrow \qquad F(x, u; \nabla_x f(u) + \nabla_p h(u, p)) - p^T \{\nabla_p h(u, p)\} \le 0$$

Definition 3.5: Let $f : R^n \times R^m \to R$ and $h : X \times R^n \to R$ be differentiable function where $X \subset R$, $F : X \times X \times R^n \to R$ be sub linear with respect to its third argument. We say that

(*i*) $f(., y)$ is higher-F-convex at $u \in X$, with respect to some function h, if for all $(x, p) \in X \times R^n$ and for fixed $y \in Y \subset R^m$ we have

$$f(x, y) - f(u, y) \ge F(x, u : \nabla_x f(u, y) + \nabla_p h(u, p))$$
$$+ h(u, p) - p^T \nabla_p h(u, p).$$

(*ii*) $f(., y)$ is said to be higher-order F-pseudo-convex at $u \in X$ with respect to h, if for fixed $y \in Y \subset R^m$ and for all $(x, p) \in X \times R^n$ we have (*iii*) $f(., y)$ is said to be higher-order F-quasi-convex at $u \in X$ with respect to h if for $(x, p) \in X \times R^n$ and for fixed $y \in Y \subset R^m$ we have

$$f(x, y) - f(u, y) \leq h(u, p) - p^T \nabla_p h(u, p)$$

$$\Rightarrow \qquad F(x, u : \nabla_x f(u, y) + \nabla_p h(u, p)) \leq 0$$

If $-f(., y)$ is higher order F convex (F pseudo-convex or F quasi-convex) at u with respect to h if for all $(x, p) \in X \times R^n$ and for fixed $y \in Y \subset R^m$ then $f(., y)$ is higher order F concave (F pseudo-concave or F quasi-concave) at u with respect to h for all $(x, p) \in X \times R^n$ and for fixed $y \in Y \subset R^m$.

Remark 1: (*i*) When $h(u, p) = \frac{1}{2} p^T \{\nabla_{uu} f(u)\} p$ and $F(x; u; a) = \eta(x, u)^T a$, where $h : X \times X \to R^n$, the higher-order F convexity (higher order F-pseudo-convexity, higher order F-quasi-convexity) reduces to η-bonvexity (η-pseudo-bonvexity, η-quasi-bonvexity) in [9].

(*ii*) When $h(u, p) = \frac{1}{2} p^T \{\nabla_{uu} f(u)\} p,$ the higher order F convexity (higher-order F-pseudo-convexity, higher order F-quasi-convexity) reduces to the second order F pseudo-invexity, F quasi-invexity in [7].

(*iii*) When $h(u, p) = \{p^T \nabla_{uu} f(u) + k(u, p)\} p$ and $F(x; u; a) = \eta(x, u)^T a,$ where $a : X \times X \to R^n_+ \setminus \{0\}, \eta : X \times X \to R^n$ are positive functions and $k : X \times R^n \to R^n$ is differentiable function, then the higher order F convexity (higher order F-pseudo convexity, higher order F-quasi-convexity) function becomes the higher-order type1 (higher order pseudo- type-1, higher order quasi-type-1) function.

From now on, suppose that the sub linear function F satisfies the following condition $F(x; y; a) + a^T y \geq 0$, for all $a \in R^n{}_+$. (1)

Definition 3.6: A real valued function $\phi(x^1, x^2, \ldots x^l)$ will be called additively separable with respect to x^1 if there exist real valued functions $\xi(x^1)$ independent of x^1, $x^2, \ldots x^l$ and $x(x^2, x^3, \ldots x^l)$ such that $\phi(x^1, x^2, \ldots x^l) = \xi(x^1) + \xi(x^2, x^3, \ldots x^l)$

4. HIGHER-ORDER SYMMETRIC DUALITY

In this section, we consider twice differentiable functions $f_i : R^n \times R^m \to R$, $g_i : R^n \times R^m \times R^{n_2} \to R$, $h_i : R^n \times R^m \times R^{m_2} \to R$ and compact convex sets $C_i \subset R^{n_2}$ and $D_i \subset R^{m_2}$ for $i = 1, 2, \ldots k$.

We formulate the following higher order symmetric nondifferentiable multiobjective Minimax mixed integer symmetric primal and dual problems.

Primal problem (MOP).

$$\max_{x^1} \max_{,x^2,y} \begin{bmatrix} (f_1(x, y) + s\langle x^2 \mid C_1 \rangle - (y^2)^T z^1 + h^1(x, y, p^1) - (p^1)^T \left[\nabla_{p^1} h^1(x, y. p^1)\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (f_k(x, y) + s\langle x^2 \mid C_k \rangle - (y^2)^T z^k + h^k(x, y, p^k) - (p^k)^T \left[\nabla_{p^k} h^k(x, y.p^k)\right] \end{bmatrix}$$

subject to

$$\sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(x, y) - z^i + \nabla_{p^i} h_i(x, y, p^i)\right] \leq 0 \qquad (2)$$

$$(y^2)^T \sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(x, y) - z^i + \nabla_{p^i} h_i(x, y, p^i)\right] \geq 0 \qquad (3)$$

$$\begin{aligned} &x^1 \in U, y^1 \in V, x^2 \in R^{n_2}, y^2 \in R^{m_2}, z^i \in D_i, p^i \in R^{m_2}, \\ &i = 1, 2, \ldots k, \lambda > 0, \lambda^T e = 1, \end{aligned} \qquad (4)$$

Dual Problem (MOD)

$$\min_{u^1} \max_{v,u^2} \begin{bmatrix} (f_1(u, v) - s\langle v^2 \mid D_1 \rangle - (u^2)^T w^1 + g_1(u, v, r^1) - (r^1)^T \left[\nabla_{r^1} g_1(u, v, r^1)\right], \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ (f_k(u, v) - s\langle v^2 \mid D_k \rangle + (u^2)^T w^k + g_k(u, v, r^k) - (r^k)^T \left[\nabla_{r^k} g_k(u, v. r^k)\right] \end{bmatrix}$$

subject to

$$\sum_{i=1}^{k} \lambda_i \left[\nabla_{u^2} f_i(u, v) - w^i + \nabla_{r^i} g_i(u, v, r^i)\right] \geq 0, \qquad (5)$$

$$(u^2)^T \sum_{i=1}^{k} \lambda_i \left[\nabla_{u^2} f_i(u, v) - w^i + \nabla_{r^i} g^i(u, v, r^i)\right] \leq 0, \qquad (6)$$

$$\begin{aligned} &u^1 \in U, v^1 \in V, u^2 \in R^{n_2}, v^2 \in R^{m_2}, w^i \in C_i, \\ &r^i \in R^{m_2}, i = 1, 2, \ldots k, \lambda > 0, \sum_{i=1}^{k} \lambda_i = 1 \end{aligned} \qquad (7)$$

Since the objective functions of (MOP) and (MOD) contain the support function $s(x^2 \mid C_i)$ and $s(v^2 \mid D_i)$, $i = 1, 2, 3, \ldots\ldots\ldots k$, they are non-differentiable multiobjective programming problems.

Remark 2: (*i*) If $U = \phi$, $V = \phi$ then (MOP) and (MOD) become the problems considered by X. Chen [15].

(*ii*) If $h_i(x, y, p^i) = \frac{1}{2}(p^i)^T \nabla_{y^2 y^2} f_i(x, y) p_i, p_i = p_i, \; g_i(u, v, r^i) = \frac{1}{2}(r^i)^T \nabla_{x^2 x^2} f_i(u, v) r_i \; r_i = r,$ and $k = 1$, then (MOP) and (MOD) can be changed into the following problems.

Primal:

$$\max_{x^1} \min_{x^2, y} [(f(x, y) + s\langle x^2 \mid C_1 \rangle - (y^2)^T z + -\frac{1}{2}(p)^T \nabla_{y^2 y^2} f(x, y) p]$$

Subject to $\nabla_{y^2} (f(x, y) - z + \nabla_{y^2 y^2} f(x, y) p \leq 0,$

$$(y^2)^T [\nabla_{y^2} f(x, y - z + \nabla_{y^2 y^2} f(x, y) p)] \geq 0,$$

$$x^1 \in U, y^1 \in V, x^2 \geq 0, z \in D, p \in R^{m - m_1}.$$

Dual:

$$\min_{v^1} \max_{v^2, u} [(f_1(u, v) - s\langle v^2 \mid D_1 \rangle + (u^2)^T w - \frac{1}{2}(r)^T \nabla_{x^2 x^2} f(u, v) r)]$$

Subject to $\nabla_{x^2}(f(u,v)+w+\nabla_{x^2x^2}f(u,v)r \geq 0,$

$$(u^2)^T[\nabla_{x^2}f(u,v)+w+\nabla_{x^2x^2}f(u,v)r] \leq 0,$$

$$u^1 \in U, v^1 \in V, v^2 \geq 0, w \in D, r \in R^{n-n_1}$$

which are the generalized forms in Hou and Yang [12].

In the sequel we shall establish the weak strong and converse duality theorem under the higher order *F*-convexity assumptions.

For this we suppose that the function $F : R^n \times R^n R^n \rightarrow R$ and $G : R^m \times R^m R^m \rightarrow R$ are sub linear and satisfy the condition

(a) $F(x,u;:a)+a^T y \geq 0, \forall a \in R_+^n,$

(b) $G(v,y:a)+a^T y \geq 0, \forall a \in R_+^n$

Also suppose that the following condition are satisfied:

(*i*) The function f_i (., v) + (.)T w_i are higher order *F*-convex at *u* with respect to $g^i(u, v, r^i)$ and

(*ii*) The function $f_i(x,.)$ + $(.)^T$ z_i are higher order *G*-convex at *y* with respect to $h^i(x, y, p^i)$ for i = 1, 2, ... k.

Theorem 4.1 (Weak Duality): Assume that and $f_i(x, y)$ and $h_i(x, y, p^i)$ are additively separable with respect to x^1 or y^1 and g_i (u, v, r^i) is additively separable with respect to u^1 or v^1. For each feasible solution (x, y, λ, z^1, z^2, ... z^k, p^1, p^2, ... p^k) of (MOP) and each feasible solution (u, v, λ, w^1, w^2, ... w^k, r^1, r^2, ... r^k) of (MOD), then the following inequality inequalities cannot hold simultaneously.

(*i*) For all $i \in$ {1, 2, 3, ... k}

$$f_i(x,y)+s(x^2\,|\,Ci)-(y^2)^T Z^i+h_i(x,y,p^i)-(p^i)^T[\nabla_{p^j}h^i(x,y,p^i)] \leq$$

$$f_i(u,v)-s(v^2\,|\,Di)+(u^2)^T w^i+g_i(u,v,r^i)-r^{iT}[\nabla_{r^i g^i}(u,v,r^i)] \text{ (A)}$$

(*ii*) For at least one $j \in \{1,2,3,...k\}$

$$f_j(x,y)+s(x^2\,|\,Cj)-(y^2)^T Z^j+h_j(x,y,p^j)-(p^j)^T[\nabla_{p^j}h^j(x,y,p^j)] \leq$$

$$f_j(u,v)s(v^2\,|\,D_j)+(u^2)^T w^j+g_j(u,v,r^j)-r^{jT}[\nabla_{p^j}h_j(x,y,,p^j)] \quad \text{(B)}$$

Proof: Since $f_i(x, y)$ and $h_i(x, y, p^i)$ are additively separable with respect x^1 or y^1 (say with respect to x^1), it holds

$$f_i(x,y) = f_{i1}(x^1) + f_{i2}(x^2,y),$$

$$h_i(x,y,p^i) = h_{i1}(x^1) + h_{i2}(x^2,y,p^i),$$

$$\nabla_{y^2} f_i(x,y) = \nabla_{y^2} f_{i2}(x^2,y)$$

and $\nabla_{p^i} h_i(x,y,p^i) = \nabla_{p^i} h_{i2}(x^2,y,p^i), i = 1,2,3.......k$

Thus (MMP) can be rewritten as

$$Z = \max_{x^1}\min_{x^2,y}\begin{bmatrix}(f_{11}(x^1)+h_{11}(x^1)+f_{12}(x^{2,},y)+s\left\langle x^2\middle|C_1\right\rangle-(y^2)^T z^1+\\ h_{12}(x^2,y,p^1)-(p^1)^T\left[\nabla_{p^1}h_{12}(x^2,y,p^1)\right],.......\\ ....,(f_{k1}(x^1)++h_{k1}(x^1)+f_{k2}(x^2,y)+s\left\langle x^2\middle|C_k\right\rangle\\ -(y^2)^T z^k+h_{k2}(x^2,y,p^k)-(p^k)^T\left[\nabla_{p^k}h_{k2}(x^2,y,p^k)\right]\end{bmatrix}$$

Subject to

$$\sum_{i=1}^{k}\lambda_i\left[\nabla_{y^2}f_{i\,2}(x^2,y)-z^i+\nabla_{p^i}h_{i2}(x^2,y,p^i)\right] \leq 0, \quad (8)$$

$$(y^2)^T\sum_{i=1}^{k}\lambda_i\left[\nabla_{y^2}f_i(x^2,y)-z^i+\nabla_{p^i}h_i(x^2,y,p^i)\right] \geq 0, \quad (9)$$

$$\begin{aligned}&x^1 \in U, y^1 \in V, x^2 \in R^{m_2}, y^2 \in R^{m_2}, z^i \in D_i, p^i \in R^{m_2},\\ &i = 1,2,.....k, \lambda > 0, \lambda^T e = 1,\end{aligned} \quad (10)$$

So (MOP) can be written as

$$Z = \max_{x^1}\min_{y^1}[(f_{11}(x^1)+h_{11}(x^1)+\phi_1(y^1),.........$$

$$......f_{k1}(x^1)+h_{k1}(x^1)+\phi_k(y^1)]$$

Subject to (8), (9) and (10) where

$$\phi_i(y^1) = \min_{x^2,y^2}\{f_{i2}(x^2,y)+h_{i2}(x^2,y,p^i)+s(x^2\,|\,C_i)$$

$$-(y^2)^T z^i-(p^i)^T\nabla_{p^i}g_{i2}(x^2,y,p^i)\}$$

Similarly, (MOD) can also be written as

$$Z^* = \min_{v^1}\max_{u^1}[(f_{11}(u^1)+h_{11}(u^1)+\psi_1(v^1),.........$$

$$......f_{k1}(u^1)+h_{k1}(u^1)+\psi_k(v^1)]$$

Subject to $\sum_{i=1}^{k}\lambda_i\left[\nabla_{u^2}f_{i2}(u^2,v)+w^i+\nabla_{r^i}g_{i2}(u^2,v,r^i)\right] \geq 0,$

(11)

$$(u^2)^T\sum_{i=1}^{k}\lambda_i\left[\nabla_{u^2}f_{i2}(u^2,v)+w^i+\nabla_{r^i}g_i(u^2,v,r^i)\right] \leq 0,$$

(12)

$u^1 \in U, y^1 \in V, u^2 \in R^{m_2}, v^2 \in R^{m_2}, w^i \in C_i, r^i \in R^{n_2}, i =$
$1,2,.....k, \lambda > 0, \lambda^T e = 1,$

(13)

For any given y^1 and v^1, the problems (MOP) and (MOD) are exactly the pair of higher order symmetric dual non-differentiable multiobjective programming problems by *X*. Chen [15]. Hence in view of the assumptions, Theorem-1 by *X*. Chen [15] becomes applicable and therefore, we have for each feasible solution

$(x,y,\lambda,z^1,z^2,..........z^k,p^1,p^2,.......p^k)$ of (MOP) and each feasible solution

$(u,v,\lambda,w^1,w^2,........w^k,r^1,r^2,.......r^k)$ of (MOD)

$$\sum_{i=1}^{k}\lambda_i\left\{(f_{i2}(u^{2,},v)-s\left\langle v^2\middle|D_i\right\rangle+(u^2)^T w^i+g_{i2}(u^2,v,r^i)-(r^i)^T[\nabla_{r^i}g_{i2}(u^2,,v,r^i)]\right\} \leq$$

$$\sum_{i=1}^{k}\lambda_i\left\{(f_{i2}(x^{2,},y)-s\left\langle x^2\middle|C_i\right\rangle+(y^2)^T z^i+h_{i2}(x^2,y,p^i)-(p^i)^T[\nabla_{p^i}h_{i2}(x^2,,y,p^i)]\right\}$$

this implies that the conclusion holds.

Remark 3:

(*i*) From now on, without loss of generality we can assume that $f_i(x, y)$, $h_i(x, y, p^i)$, and $g_i(x, y, r^i)$ are additively separable with respect to x^i, $i = 1, 2, . k$

(*ii*) From the process of the proof in theorem-1, we can also obtain that (*A*) and (*B*) cannot hold simultaneously if sub linear functions *F* and *G* satisfy the condition (*a*) and (*b*) and for each feasible solution $(x, y, \lambda, z^1, z^2, \dots z^k, p^1, p^2, \dots p^k)$ of (MOP) and each feasible solution $(u, v, \lambda, w^1, w^2, \dots w^k, r^1, r^2, \dots r^k)$ of (MOD), one of the following conditions holds.

(*i*) $f_i(u^1, ., v) + (.)^T w_i$ is higher order *F*-pseudo convex at u^2 with respect to $g_{i2}(u^2, v, r^i)$ and $f_i(x, y^1, .) - (.)^T z_i$ is higher – order G-pseudo-concave at y^2 with respect to $h_{i2}(x^2, y, p^i)$.

(*ii*) $f_i(u^1, .., v) + (.)^T w_i$ is higher order *F*-quasi-convex at u^2 with respect to $g_{i2}(u^2, v, r^i)$ and $f_i(x, y^1,.) - (.)^T z^i$ is higher – order *G*-quasi-concave at y^2 with respect to $h_{i2}(x^2, y, p^1)$.

The following result indicates that under some conditions, a properly efficient solution of (MOP) is also the ones of (MOD) and the two objective values are correspondingly equal.

Theorem 4.2 (Strong Duality):

Let $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, \dots \bar{z}^k, \bar{p}^1, \bar{p}^2, \dots \bar{p}^k)$ be an efficient solution of (MMP), $f_i : R^n \times R^m \to R$ is twice differentiable at $(\bar{x}, \bar{y})$, $h_{i2} : R^{n_2} \times R^m \times R^{m_2} \to R$, $g_{i2} : R^{n_2} \times R^m \times R^{n_2} \to R$, is twice differentiable at $(\bar{x}, \bar{y}, \bar{p}^i)$, is twice differentiable at $(\bar{x}, \bar{y}, \bar{p}^i)$, for *i* = 1, 2, 3, ... *k*. Assume that the following conditions hold;

(*i*) $h_{i2}(\bar{x}^2, \bar{y}, 0) = 0, g_{i2}(\bar{x}^2, \bar{y}, 0) = 0, \nabla_{p^i} h_{i2}(\bar{x}^2, \bar{y}, 0) = 0, \nabla_y h_{i2}(\bar{x}^2, \bar{y}, 0) = 0$

$$\nabla_{x^2} h_{i2}(\bar{x}^2, \bar{y}, 0) = \nabla_{r^i} h_{i2}(\bar{x}^2, \bar{y}, 0), i = 1,2,3....k;$$

(*ii*) for all $i \in \{1, 2, 3, ...k\}$, we have the Hessian matrix $\nabla_{p^i p^i} h_{i2}(\bar{x}^2, \bar{y}, \bar{p}^i)$, is positive definite or negative definite;

(*iii*) the set of vectors

$$\left\{\nabla_{y^2} f_{i2}(\bar{x}^2, \bar{y}) - \bar{z}^i + \nabla_{p^i} h_{i2}(\bar{x}^2, \bar{y}, \bar{p}^i)\right\}_{i=1}^{k}$$

is linearly independent,

(*iv*) For some $a \in R^k (a > 0)$ and $p^i \in R^{m_2}, p^i \neq 0$, $i = 1,2,3,...k$ we have

$$\sum_{i=1}^{k} a_i (p^i)^T \left[\nabla_{y^2} f_{i2}(\bar{x}^2, \bar{y}) - \bar{z}^i + \nabla_{p^i} h_{i2}(\bar{x}^2, \bar{y}, \bar{p}^i)\right] \neq 0.$$

Then $\bar{p}^i = 0, i = 1,2,3....k;$ And there exists $\bar{w}^i \in C_i$ such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}^1, \bar{w}^2, ...\bar{w}^k, \bar{r}^1 = 0, \bar{r}^2 = 0....\bar{r}^k = 0)$ is a feasible solution of (MOD). Furthermore, if the hypotheses in Theorem 3.1 are satisfied and $h_{i1}(\bar{x}^i) = g_{i1}(\bar{x}^i)$, *i* = 1, 2, 3, ... *k*, then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}^1, \bar{w}^2, ...\bar{w}^k, \bar{r}^1 = 0, \bar{r}^2 = 0....\bar{r}^k = 0)$ is an efficient solution of MOD), and the two objective values are equal.

Proof: If $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, \dots \bar{z}^k, \bar{p}^1, \bar{p}^2, \dots \bar{p}^k)$ a properly efficient for (MMP) then $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, \dots \bar{z}^k, \bar{p}^1, \bar{p}^2, \dots \bar{p}^k)$ is also efficient for (MP). Thus, under the condition in this theorem we obtain from theorem-2 in *X*. Chen[15] that there exist $\bar{w}^i \in C_i, i = 1,2,3,.......k$, such that $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{w}^1, \bar{w}^2, ... \bar{w}^k, \bar{r}^1 = \bar{r}^2 = ... = \bar{r}^k) = 0$ is a feasible solution of (MD). It is obvious that it is also feasible for (MMD).

Furthermore, if the hypotheses of higher order F-convexity in theorem-1 are satisfied, then the objective values of (MP) and (MD) are equal by [15], that is

$$f_{i2}(\bar{x}^{2,}, \bar{y}) + s\langle \bar{x}^2 | C_i \rangle - (\bar{y}^2)^T \bar{z}^i + h_{i2}(\bar{x}^2, \bar{y}, \bar{p}^i) - (p^i)^T [\nabla_{p^i} h_{i2}(\bar{x}^2,, \bar{y}, \bar{p}^i)]$$
$$= (f_{i2}(\bar{x}^{2,}, \bar{y}) - s\langle \bar{y}^2 | D_i \rangle + (\bar{x}^2)^T w^i + g_{i2}(\bar{x}^2, \bar{y}, \bar{r}^i) - (\bar{r}^i)^T [\nabla_{r^i} g_{i2}(\bar{x}^2, \bar{y}, \bar{r}^i)]$$

i = 1, 2, 3, 4 …….. *k*.

Note that $h_{i1}(\bar{x}^i) = g_{i1}(\bar{x}^i), \nabla_{p^i} h_{i1}(\bar{x}^1) = 0, \nabla_{r^i} g_{i1}\ (\bar{x}^1) = 0$, we have

$$f_i(\bar{x}', \bar{y}) + s\langle \bar{x}^2 | C_i \rangle - (\bar{y}^2)^T \bar{z}^i + h_i(\bar{x}^2, \bar{y}, \bar{p}^i) - (p^i)^T [\nabla_{p^i} h_i(\bar{x}, \bar{y}, \bar{p}^i)]$$
$$= (f_i(\bar{x}, \bar{y}) - s\langle \bar{y}^2 | D_i \rangle + (\bar{x}^2)^T \bar{w}^i + g_i(\bar{x}, \bar{y}, \bar{r}^i) - (\bar{r}^i)^T [\nabla_{r^i} g_i(\bar{x}, \bar{y}, \bar{r}^i)], i = 1,2,...k$$

From theorem-1, $(\bar{x}^2, \bar{y}, \bar{\lambda}, \bar{w}^1, \bar{w}^2, \dots \bar{w}^k, \bar{r}^1 =, \bar{r}^2 = \cdots = .\bar{r}^k) = 0$ is an efficient solution of (MMD).

It is similar to the method of the proof of theorem – 2 in *X*. Chen-2004[15] that it is also a properly efficient solution of (MMD). Similarily, we have the following converse Duality.

Theorem 4.3: (Converse Duality): Let $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{w}^1, \bar{w}^2, ... \bar{w}^k, \bar{r}^2 = ... = .\bar{r}^k) = 0$ be a properly efficient solution of (MMD), $f_i : R^n \times R^m \to R$ is twice differentiable at $(\bar{u}, \bar{v})$, $g_i : R^n \times R^m \times R^{n_2} \to R$ is twice differentiable at $(\bar{u}, \bar{v}, \bar{r}_i)$, $h_i : R^n \times R^m \times R^{n_2} \to R$ is differentiable at $(\bar{u}, \bar{v}, \bar{r}_i)$ if the following conditions hold

(*i*) $h_{i2}(\bar{u}^2, \bar{v}, 0) = 0, g_{i2}(\bar{u}^2, \bar{v}, 0) = 0, \nabla_{r^i} g_{i2}(\bar{u}^2, \bar{v}, 0) = 0$, $\nabla_{y^2} g_{i2}(\bar{u}^2, \bar{v}, 0) = \nabla_{p^i} h_{i2}(\bar{u}^2, \bar{v}, 0), i = 1,2,3,...k;$

(*ii*) For all $i \in \{1, 2, 3, ...k\}$, the Hessian matrix $\nabla_{r^i r^2} g_{2i}\ (\bar{u}^2, \bar{v}, \bar{r}^i)$ is positive definite or negative definite.

(*iii*) The set of vectors

$$\left\{\nabla_{x^2} f_{i2}\ (\bar{u}^2, \bar{v}) + \bar{w}^i + \nabla r^i g_{i2}(\bar{u}^2, \bar{v}, \bar{r}^i)\right\}_{i=1}^{k}$$

is linearly independent.

(iv) For some $\alpha \in R^k (\alpha > 0)$ and $r^i \in R^{n_2}, r^i \neq 0$ $(i = 1,2,3,...k)$ implies that

$$\sum_{i=1}^{k} \alpha_i r^{iT} \left\{ \nabla_{x^2} f_{i2}(\bar{u}^2, \bar{v}) + \bar{w}^i + \nabla_{r^i} g_{i2}(\bar{u}^2, \bar{v}, \bar{r}^i) \right\} \neq 0$$

Then (i) $\bar{r}^i = 0, i = 1,2,3,...k;$

(ii) There exists $\bar{z}^i \in C_i$ such that $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}^1, \bar{z}^2,, ...\bar{z}^k, \bar{r}^1 =, \bar{r}^2 = ... = .\bar{r}^k) = 0$ is feasible solution of (MMP).

Furthermore, if the hypotheses in Theorem 3.1 are satisfied and $g_{i1}(\bar{x}^1) = h_{i1}(\bar{x}^1), i = 1, 2, ...k$ then $(\bar{u}, \bar{v}, \bar{\lambda}, \bar{z}^1, \bar{z}^2,, ...\bar{z}^k, \bar{r}^1 =, \bar{r}^2 = ... = .\bar{r}^2) = 0$ is properly efficient solution of (MMP), and the two objective values are correspondingly equal.

5. HIGHER-ORDER SELF DUALITY

A mathematical programming problem is said to be self-dual, if when the primal is recast in the form of the dual, the new problem obtained is the same as the dual problem.

First, we give the following definition.

Definition 5.1: The function $h : I \times R^{n_1} \times R^{n_2} \times R^{n_1} \times R^{n_2} \to R$ is said to be skew-symmetric with respect to x and y if for all x and y in the domain of h such that $h(x^1, x^2, y^1, y^2) = -h(y^1, y^2, x^1, x^2)$ where $x^1 \in U$, $x^2 \in R^{n_2}$ and $y^2 \in R^{n_2}$ and U is an arbitrary sets of integers in $R^{n_1}, n_1 + n_2 = n$

Theorem 5.1 (Self-duality): If f_i and h_i in (MMP) are skew symmetric function with respect to x and y and $m = n$, $U = V$, $C_i = D_i$, $z^i = w^i$, $p^i = r^i$ and $h(x, y, p^i) = g(x, y, r^i)$, $i = 1, 2, 3, ... k$. Then (MMP) is self-dual, that is, the dual problem of (MMP) is itself, and the proper efficiency of $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, ...\bar{z}^k, \bar{p}^1, \bar{p}^2, ...\bar{p}^k)$ for (MMD), and the converse. Furthermore, under the conditions of theorem (4.2) and (4.3), if $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, ...\bar{z}^k, \bar{p}^1 = \bar{p}^2 = ...\bar{p}^k = 0)$ is a properly efficient solution of (MMP), then $(\bar{y}, \bar{x},, \bar{\lambda}, \bar{z}^1, \bar{z}^2, ...\bar{z}^k, \bar{r}^1 = \bar{r}^2 = ...\bar{r}^k = 0)$ aproperly efficient solution of (MMD), the common optimal values is zero and the onverse.

Proof: The problem (MMP)may be represented as a max-min problem

$$\max_{x,^1 x^2, y, z^i, p^i} \min \begin{bmatrix} -(f_1(x,y) - s\langle x^2 | C_1 \rangle + (y^2)^T z^1 - h^1(x,z,p^1) + (p^1)^T \left[\nabla_{p^1} h_1(x,z.p^1)\right], \\ ......, -(f_k(x,y) - s\langle x^2 | C_k \rangle + (y^2)^T z^k - h^k(x,z,p^k) + (p^k)^T \left[\nabla_{p^k} h_k(x,z.p^k)\right], .. \end{bmatrix}$$

subject to $\sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(x,,y) - z^i + \nabla_{p^i} h_i(x,y,p^i) \right] \leq 0,$

$$(y^2)^T \sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(x,y) - z^i + \nabla_{p^i} h_i(x,y,p^i) \right] \geq 0,$$

$$x^1 \in U, y^1 \in V, x^2 \in R^{n_2}, y^2 \in R^{m_2}, z^i \in D_i,$$

$$p^i \in R^{m_2}, i = 1,2,.....k, \lambda > 0, \lambda^T e = 1,$$

Since f_i and h_i is skew-symmetric function with respect to x and y $C_i = D_i$, $z^i = w^i$, $p^i = r^i$ and $h(x, y, p^i) = g(x, y, p^i) = g(x, y, r^i)$, $i = 1, 2, 3, ... k$. it holds

$$\min_{x,^1 x^2, y, z^i, p^i} \max \begin{bmatrix} (f_1(y,x) - s\langle x^2 | D_1 \rangle + (y^2)^T w^1 + g_1(y,x,r^1) - (r^1)^T \left[\nabla_{r^1} g^1(y,x,r^1)\right],, \\ ......, -(f_k(y,x) - s\langle x^2 | D_k \rangle + (y^2)^T w^k + g_k(y,x,r^k) - (r^k)^T \left[\nabla_{k^k} g_k(y,x.r^k)\right], .. \end{bmatrix}$$

Subject to $\sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(y,x) + w^i + \nabla_{r^i} g_i(y,x,r^i) \right] \geq 0,$

$$(y^2) \sum_{i=1}^{k} \lambda_i \left[\nabla_{y^2} f_i(y,x) + w^i + \nabla_{r^i} g_i(y,x,r^i) \right] \leq 0,$$

$x^1 \in U, y^1 \in V, x^2 \in R^{n_2}, y^2 \in R^{n_2}, w^i \in D_i, \quad r^i \in R^{n_2}, i = 1, 2,.....k, \lambda > 0, \lambda^T e = 1,$ which is the dual problem (MMD). Thus (MMP) is self –dual. It is obvious that the proper efficiency of $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, ...\bar{z}^k, \bar{p}^1, \bar{p}^2, ...\bar{p}^k)$ for (MMP) implies the proper efficiency of $(\bar{x}, \bar{y}, \bar{\lambda}, \bar{z}^1, \bar{z}^2, ...\bar{z}^k, \bar{p}^1, \bar{p}^2, ...\bar{p}^k)$ for (MMD),and the converse.

Next, we show that

$$f_i(\bar{x}', \bar{y}) + s\langle \bar{x}^2 | C_i \rangle - (\bar{y}^2)^T \bar{z}^i + h_i(\bar{x}^2, \bar{y}, \bar{p}^i)$$
$$-(p^i)^T [\nabla_{p^i} h_i(\bar{x}, \bar{y}, \bar{p}^i)] = 0$$

$i = 1, 2, 3, \ldots k \ldots(18)$

By theorem (4.2), (4.3) and (5.1) we have

$$\begin{aligned} & f_i(\bar{x}', \bar{y}) + s\langle \bar{x}^2 | C_i \rangle - (\bar{y}^2)^T \bar{z}^i + h_i(\bar{x}^2, \bar{y}, \bar{p}^i) - (p^i)^T [\nabla_{p^i} h_i(\bar{x}, \bar{y}, \bar{p}^i)] \\ & = (f_i(\bar{y}, \bar{x}) - s\langle \bar{x}^2 | D_i \rangle + (\bar{y}^2)^T w^i + g_i(\bar{y}, \bar{x}, \bar{r}^i) - (\bar{r}^i)^T [\nabla_{r^i} g_i(\bar{y}, \bar{x}, \bar{r}^i)] \\ & = -f_i(\bar{x}', \bar{y}) - s\langle \bar{x}^2 | C_i \rangle + (\bar{y}^2)^T \bar{z}^i - h_i(\bar{x}^2, \bar{y}, \bar{p}^i) + (p^i)^T [\nabla_{p^i} h_i(\bar{x}, \bar{y}, \bar{p}^i)] \end{aligned}$$

Where the equality is from the conditions. This implies that (18) holds.

6. CONCLUSION

In the above, we formulate a pair of the higher-order symmetric non-differentiable multiobjective min-max mixed programming problem in which the objective functions contain a support function of a compact convex set in R^n or R^m.

Under the higher-order F-convexity (higher-order F-pseudo-convexity, higher-order F-quasi-convexity) assumption, we give the higher-order weak., higher-order strong, higher-order converse duality, and self duality. In our models, $U = \phi, V = \phi,$ then (MMP) and (MMD) become the problems considered by X. Chen[15]. If $h_i(x, y, p)$ = $\frac{1}{2} p^T \nabla_{yy} f_i(x,y) p \; g_i(u, v, r) = \frac{1}{2} r^T \nabla_{uu} f_i(u,$ $v)r$ and $k = 1$, $U = \phi$, $V = \phi$ then (MMP) and (MMD) reduce to the second-order symmetric models of Hou and Yang[12].

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