

ON THE USE OF COMPUTERS IN GRAPH LABELING

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ABSTRACT

The major vertex labellings (Harmonious, Sequential, Felicitous, Graceful, Antimagic, Magic) present many labelling problems to be solved for various classes of graphs. Manual work can take an enormously long time and also does not allow much possibility for experimentation with unconventional ideas. Algorithms and a specialized software have been developed to check existence of these major graph labelling schemes for an arbitrary graph and make possible such experimentation and exploring possibilities which otherwise would not be possible manually. This paper discusses the results obtained using this software, works done by different authors in graph labellings using computer and presents some new findings.

Keywords: vertex label, edge label, algorithm

1. INTRODUCTION

Regarding the use of computers in graph labellings, Aldred and McKay (preprint,1998) have used computer to show that trees with at most 26 vertices are Harmonious. Eshghi and Azimi(2004) give programming model to find graceful labelling of graphs. Eshghi give and Azimi(2007) give an algorithm to find solutions of graph labelling problems. They have used their model to show that trees with 30, 35 or 40 vertices are graceful. Krishnaa(2001) has developed algorithms and a generalized computer software to test for existence of any of the major labelling schemes viz., Harmonious, Sequential, Graceful, Felicitous, Antimagic and Magic for an arbitrary graph alongwith the option in the software to test the behaviour of graphs when a single vertex label is repeated for certain labellings. The results obtained by the computer software developed by Krishnaa (2001) are being discussed in greater detail since the results obtained by this software are several and varied and shed light on the various possibilities which become possible to explore by the use of computer.

Algorithms and computer programs (in C programming language) have been developed to generate permutations and combinations of numbers in lexicographic order; this order has been mentioned in Liu(1978). In lexicographic order, the word 'cart' will appear before 'carp' as in a dictionary or lexicon. This concept can be extended to the sequence of numbers also where if 1 2 3 4 is taken as the first permutation then the next or second permutation in lexicographic order is 1 2 4 3. In the computer programs, for testing for graphs, the lexicographic order may be used to generate permutations and combinations exhaustively or in the case if the graph is large then any particular

sequence can be given as the starting sequence and stopping when one wants to.

In Krishnaa(2001), three algorithms have been developed which have served the basis for developing the computer software for testing an arbitrary graph for existence of any of the major kinds of labellings namely Harmonious, Felicitous, Sequential, Graceful, Magic and Antimagic.

2. DISCUSSION OF RESULTS

Let us look at the 24 permutations for 4! permutations in lexicographic order as follows:

1234, 1243, 1324, 1342, 1423, 1432,
 2134, 2143, 2314, 2341, 2413, 2431,
 3124, 3142, 3214, 3241, 3412, 3421,
 4123, 4132, 4213, 4231, 4312, 4321.

The 15 combinations of 6C_4 in lexicographic order are as follows :

1234, 1235, 1236, 1245, 1246, 1256, 1345, 1346,
 1356, 1456,
 2345, 2356,
 3456.

The algorithms namely PERM and COMB generate permutations and combinations in lexicographic order and are based on the concept given in Liu(1978). These algorithms are given as follows :

Algorithm PERM :

Let $A = \{a_1, a_2, a_3, \dots, a_n\}$, $n = |A|$, $1 \leq m < n$; enumerate all the factorial(n) permutations

1. Denote the starting permutation as $a_1, a_2, a_3, \dots, a_n$.
 Set num_perm = 0

2. Go from right to left and make note of $a_m < a_{m+1}$ where am is the rightmost such element
3. Go from right to left to select a_p where a_p is the rightmost element satisfying

$$a_m < a_p$$

4. Interchange am and ap and get the current permutation $a_1, a_2, a_3, \dots, a_n$
5. Initialise $l = 0$
while $(m + i) \leq (n - i)$
 $i = i + 1$
if $(m + i) < (n - i + 1)$ (subscripts of m and p are from step 2 and step 3)
interchange the values of a_{n-i+1} and a_{m+i}
endif
6. The resulting new $a_1, a_2, a_3, \dots, a_n$ is the next permutation
7. num_perm = num_perm+1
8. Continue steps 1 through 7 for factorial $(n - 1)$ times.

Algorithm COMB:

Let the starting combination be $a_1, a_2, a_3, \dots, a_r$; ${}^n C_r$ combinations are computed.

1. Starting combination is $a_1, a_2, a_3, \dots, a_r$
2. Set flag = no. of consecutive integers $n, n - 1, \dots$ from right to left e.g., in 1256, flag = 2.
3. if flag < 2
choose the largest m from left to right such that $am < n$
and make note of m .
endif
- if flag ≥ 2 then
move from right to left till the consecutive integers are over and these
select $am < n$ and make note of m .
endif
4. $b_m = a_m + 1$
5. If $m < r$ then do the following loop
 $for j = m \text{ to } (r - 1)$
 $b_{j+1} = b_j + 1$
endifor
6. Next combination in lexicographic order is $a_1, a_2, a_3, a_{m-1}, b_m, b_{m+1}, \dots, b_r$.
7. Continue steps 1 through 5 for ; ${}^n C_r$ times.

Another algorithms called REPEAT-ONE has been developed to generate numbers so that a single number is repeated. This algorithm will generate numbers as follows for the set $\{0,1,2,3,4\}$:

01230, 01204, 01304, 00234,
11234, 01214, 01134, 01231,
21234, 01224, 01232, 02234,
312334, 01233, 01334, 03234,
41234, 01244, 01434, 04234.

Algorithms REPEAT-ONE:

The original array is $A = \{0, 1, 2, \dots, q\}$
for ext = 0 to q // the variable ext is the repeated value //
for $j = q$ to 1, step (-1)
if $(j \bmod q) = \text{ext}$ then
begin-if
 $A(j \bmod q) = \text{ext}$
 $A(q) = \text{ext}$
endif
if $(j \bmod q) \neq \text{ext}$ then
 $A(j \bmod q) = \text{ext}$
endif
in either case do the following for-loop
for $k = 0$ to q
 $B(k) = A(k)$ // array B stores the sequence with a repeated value //
endifor (j)
endifor (ext)

Depending on the capacity of the computer memory and speed of the microprocessor, it yields several labellings in various labelling schemes when they exist. Using this software, Krishnaa, Dulawat and Rathore (2004) have studied the Antimagic labelling for certain classes of graphs namely Wheels, Cycles, Stars, Complete graphs and Paths and the effect of degree of vertex on obtaining the number of Antimagic labellings. A large number of Antimagic labellings have been obtained for all the classes of graphs studied and shows that the presence of higher degree vertices results in higher number of Antimagic labellings obtained. It may be noted that the number of Antimagic labellings obtained is in hundreds and even in thousands. Statistical analysis also verifies the results. Following tables show the results obtained:

Table1
Number of Antimagic Labellings for Various Graphs

Edges	No. of permutations Tested for	Path	Cycle	Wheel	Star
6	50	11	10	19	50
8	700	93	78	428	700
10	5000	407	346	2825	5000
12	10000	786	845	3574	10000
14	25000	1538	1670	6196	25000

Now, let us consider the role played by the degree of vertices in graphs closely related in terms of degree of vertex in yielding the number of Antimagic labellings. A Path has all vertices of degree two except two end vertices which have degree one whereas a Cycle has all the vertices of degree two. These two graphs have been comparatively studied for the relationship between the

number of Antimagic labellings and the number of permutations tried. For statistical analysis the hypotheses formulated are as follows :

$$H_0 : \text{Mean of Cycles (Mean of Antimagic labellings)} \\ = \text{Mean of Paths}$$

$$H_1 : \text{Mean of Cycles} > \text{Mean of Paths}$$

Table 2
Analysis of Antimagic Labellings for Paths and Cycles

No of edges	No. of permutations tested for	Path	Cycle	Computations for tests	Tablaated values
3	6	2	5		
4	24	5	8		
5	120	16	29		
6	720	91	96	1 tail, equal var, calculated $t = 0.08738$	1 tail at 5% $Df = 18$, tabulated $t = 1.7341$
7	1000	118	135		
8	2000	190	206		
9	3000	257	267		
10	5000	407	346		
12	10000	786	845		
14	25000	1538	1670		

Applying the t-test, the calculated value is less than the tabulated value for the t-table for 18 degrees of freedom and hence well within the acceptance region and we retain the null hypothesis H_0 . Thus the hypotheses represent the true situation and the results obtained show no significant difference.

A more complex structure in terms of degree of vertex is Wheel, which is obtained by adding an internal vertex to the Cycle. This results expectedly in a much higher number of Antimagic labellings obtained than the Path-Cycle data and shown in the following table :

Table 3
Number of Antimagic Labellings obtained for Cycles and Wheels

No of edges	No. of permtations tested for	Cycle	Wheel	Computations for tests	Tablaated values
6	50	10	19		1 tail, $df = 8$
8	700	78	428	1 tail, equal var, calculated $t = 2.635$	At 5% tabulated $t = 1.8595$
10	5000	346	2825		
12	10000	845	3574		
14	25000	1670	6196		

The hypotheses are as follows :

$$H_0 : \text{Mean of Wheels (Mean of Antimagic labellings)} \\ = \text{Mean of Cycles}$$

$$H_1 : \text{Mean of Wheels} > \text{Mean of Cycles}$$

The calculated value is more than the tabulated value and we reject the null hypothesis and the results obtained

also support this.

Therefore, we see an increase in the number of Antimagic labellings obtained as the presence of higher degree vertices increases. This conclusion is further strengthened by the labellings obtained for the Complete graphs which have the highest degree of vertices.

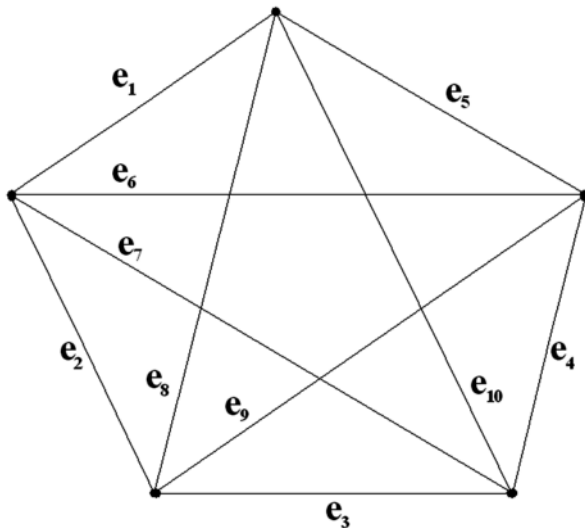


Figure 1: Complete Graph K_5

For the Complete graphs, K_4 gives 240 Antimagic labellings for 720 permutations tried, whereas K_5 gives 1682 Antimagic labellings for 3000 permutations tried which is a very high number. The edges e_1, e_2, e_3, \dots as shown in the figure 1 are labeled using the software and some of these results are shown as follows :

Testing for 3000 permutations of the edge labels:

Labelling num 1

Permutation 4 of edge labels : 1 2 3 4 5 6 7 9 10 8

Respective vertex labels on v_1, v_2, v_3 etc : 16, 24, 22, 25, 23

Labelling num 2

Permutation 7 of edge labels : 1 2 3 4 5 6 8 7 9 10

Respective vertex labels on v_1, v_2, v_3 etc : 17 21 25 24 23

Labelling num 1681

Permutation 2998 of edge labels: 1 2 3 8 4 10 9 6 7 5

Respective vertex labels on v_1, v_2, v_3 etc : 22 18 25 29 16

Labelling num 1682

Permutation 2999 of edge labels: 1 2 3 8 4 10 9 7 5 6

Respective vertex labels on v_1, v_2, v_3 etc : 22 17 26 27 21

Therefore, a definite relation between degree of vertices and number of Antimagic labellings is obtained with the presence of higher degree vertices yielding a higher number of Antimagic labellings in a graph. This finding is also supported by statistical analysis.

Krishnaa and Dulawat (2009) have studied the effect and presence of lexicographic order of permutations in labellings of certain classes of graphs like Cycles, Paths and Complete Bipartite graphs. Some new findings are also being presented here which have not been discussed in the above mentioned paper by Krishnaa and Dulawat (2009). The results obtained for Paths are shown in figure 2.

The permutation 1 gives Antimagic labelling of Paths with even number of vertices whereas the permutation 2 gives Antimagic labelling for Paths with odd number of vertices.

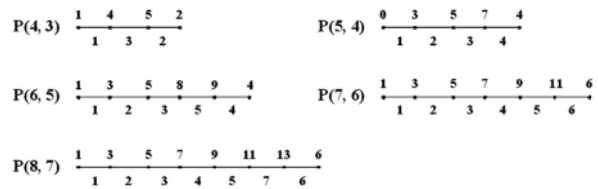


Figure 2: Antimagic Paths; Permutation 1 for odd vertices, Permutation 2 for even vertices

Let us look at the computer generated results for the Cycles where the edges e_1, e_2, e_3, \dots are labeled in the order of the permutations shown in figure 3.

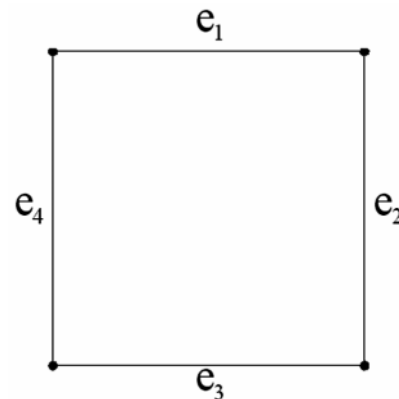


Figure 3: Cycle

Even Cycles :

C4 :

Permutation 2: 1 2 4 3

Vertex labels (in ascending order) : 3 4 6 7

C6 :

Permutation 2: 1 2 3 4 6 5

Vertex labels (in ascending order) : 3 5 6 7 10 11

C8 :

Permutation 2: 1 2 3 4 5 6 8 7

Vertex labels (in ascending order) : 3 5 7 8 9 11 14 15

C10 :

Permutation 2: 1 2 3 4 5 6 7 8 10 9

Vertex labels (in ascending order) : 3 5 7 9 10 11 13 15 18 19

Now, let us look at the odd Cycles :

C5 :

Permutation 1: 1 2 3 4 5

Vertex labels (in ascending order) : 3 5 6 7 9

C_7 :

Permutation 1: 1 2 3 4 5 6 7

Vertex labels (in ascending order) : 3 5 7 8 9 11 13

Permutation 3: 1 2 3 4 6 5 7

Vertex labels (in ascending order) : 3 5 7 8 10 11 12

C_9 :

Permutation 1: 1 2 3 4 5 6 7 8 9

Vertex labels (in ascending order) : 3 5 7 9 10 11 13 15 17

Permutation 3: 1 2 3 4 5 6 8 7 9

Vertex labels (in ascending order) : 3 5 7 9 10 11 14 15 16

C_{11} :

Permutation 1: 1 2 3 4 5 6 7 8 9 10 11

Vertex labels (in ascending order) : 3 5 7 9 11 12 13 15 17 19 21

Permutation 3: 1 2 3 4 5 6 7 8 10 9 11

Vertex labels (in ascending order) : 3 5 7 9 11 12 13 15 18 19 20

In Cycles, permutation 1 gives Antimagic labelling for odd Cycles whereas permutation 2 gives Antimagic labelling for even Cycles. Permutation 3 also gives Antimagic labelling for odd Cycles except C_5 . Permutation 1 results in induced vertex labels showing almost an arithmetic progression with difference 2 as well.

Let us look at the Complete Bipartite graphs shown in figure 4.

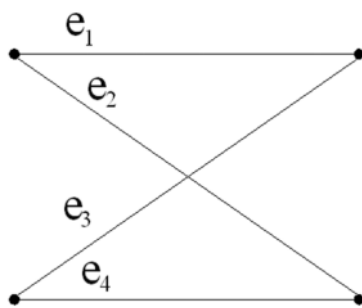


Figure 4: Complete Bipartite Graph

Some new findings are presented here which have not been discussed in the above mentioned paper by Krishnaa and Dulawat (2009). We look at the following results for $K_{n,n}$ when n is even and with a lexicographic order with permutation 1. In the left partite set, edges e_1, e_2, e_3, \dots are numbered with $1, 2, \dots, n \times n$ top to bottom with permutation 1. This labelling results in the Antimagic labelling of the $K_{n,n}$ when n is even. The following results are obtained :

$K_{2,2}$: Induced vertex labels of the left partite set from top to bottom : 3, 7.

Induced vertex labels of the right partite set from top to bottom : 4, 6.

A difference of 4 in the vertex labels of the left partite set and a difference of 4 in those of the right partite set is observed.

$K_{4,4}$: Induced vertex labels of the left partite set from top to bottom : 10, 26, 42, 58.

Induced vertex labels of the right partite set from top to bottom : 28, 32, 36, 40.

We observe a difference of 26 in the induced vertex labels of left partite set and a difference of 4 in the induced vertex labels of the right partite set.

$K_{6,6}$: Induced vertex labels of the left partite set from top to bottom : 21, 57, 93, 129, 165, 201.

Induced vertex labels of the right partite set from top to bottom : 96, 102, 108, 114, 120.

We observe a difference of 36 in the two successive induced vertex labels of left partite set and a difference of 6 in the induced vertex labels of the right partite set.

$K_{8,8}$: Induced vertex labels of the left partite set from top to bottom : 36, 100, 164, 228, 292, 356, 420, 484.

Induced vertex labels of the right partite set from top to bottom : 232, 240, 248, 256, 264, 272, 280, 288.

We observe a difference of 64 in the two successive induced vertex labels of left partite set and a difference of 8 in the induced vertex labels of the right partite set.

$K_{10,10}$: Induced vertex labels of the left partite set from top to bottom : 55, 155, 255, 355, 455, 555, 655, 755, 855, 955.

Induced vertex labels of the right partite set from top to bottom : 460, 470, 480, 490, 500, 510, 520, 530, 540, 550.

We observe a difference of 100 in the two successive induced vertex labels of left partite set and a difference of 10 in the induced vertex labels of the right partite set.

Therefore, we see that there is much regularity in the Antimagic labelling obtained for the Complete Bipartite graph tried for $K_{n,n}$ when n is even where there is a fixed difference in the induced vertex labels in the left partite set and the difference in the induced vertex labels obtained in the right partite set forms an arithmetic progression of the differences namely 2,4,6,8,... Therefore, in the cases where the graphs show certain regularity in degree of vertices i.e., most vertices are of same degree, the lexicographic order gives significant results.

Krishnaa, Dulawat and Rathore (2005) have studied the patterns obtained in graph labellings, some of these results are presented here.

The Cycles C_7, C_9, C_{11} also demonstrate an alternate sequence in labelling of the vertices v_1, v_2, v_3, \dots as follows :

C_5 : 1 4 2 5 3

C_7 : 1 5 2 6 3 7 4

$C_9 : 1\ 6\ 2\ 7\ 3\ 8\ 4\ 9\ 5$

$C_{11} : 1\ 7\ 2\ 8\ 3\ 9\ 4\ 10\ 5\ 11\ 6$

Resulting induced edge labels show much regularity in patterns in the graph also with consecutive induced edge labels obtained as well for the edges as shown below for C_7 in figure 5.

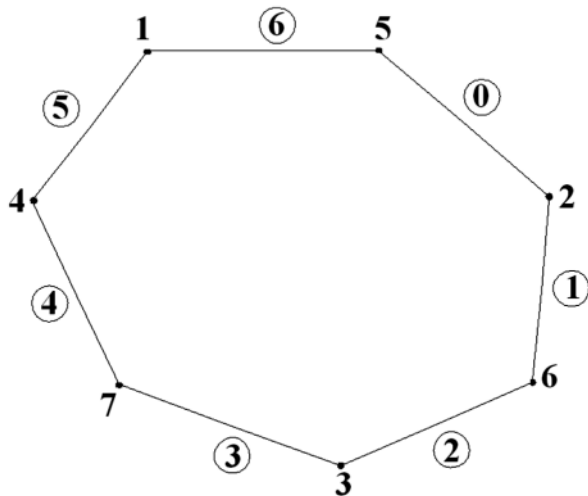


Figure 5: Felicitous Cycle C_7 with Alternate Vertex Labels

In odd Cycles, permutation 1 yields Felicitous labelling as well. Labelling the vertices with permutation 1 of the Cycle in figure 2 yields the induced Felicitous edges in the following way with zero in the middle with two arithmetic progressions on each side of the graph as shown in figure 6.

$C_5 : 1\ 3\ 0\ 2\ 4$

$C_7 : 1\ 3\ 5\ 0\ 2\ 4\ 6$

$C_9 : 1\ 3\ 5\ 7\ 0\ 2\ 4\ 6\ 8$

$C_{11} : 1\ 3\ 5\ 7\ 0\ 2\ 4\ 6\ 8\ 10$

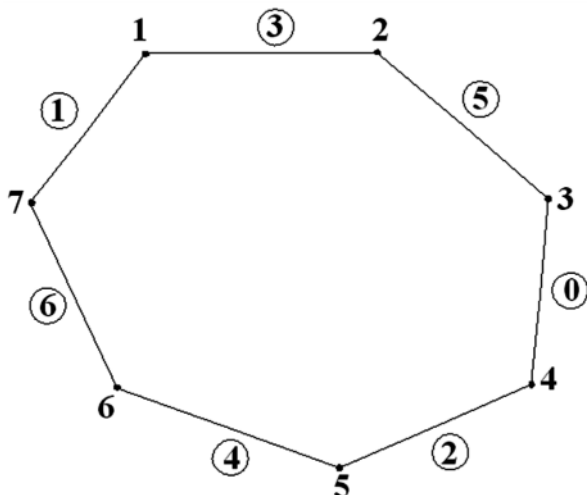


Figure 6: Felicitous Cycle C_7 with Permutation 1 of Vertex Labels

Observing the patterns for repeating a vertex label also were done. The Trees shown in figure 6, $T(7, 6)$ and $T(10, 9)$ did not produce any Graceful labelling for a great number of permutations but produced Graceful labelling much quicker and several in number when a vertex label was repeated. The vertex labels are numbered with following labels in the order of v_1, v_2, v_3, \dots as shown in figure 7.

$T(7,6) : 0\ 2\ 0\ 6\ 1\ 3\ 5, 0\ 4\ 0\ 1\ 3\ 2\ 5, 0\ 5\ 0\ 1\ 4\ 3\ 2, 1\ 1\ 3\ 5\ 6\ 0\ 4, 0\ 3\ 0\ 2\ 1\ 4\ 5, 0\ 5\ 0\ 1\ 4\ 2\ 3$

$T(10,9) : 0\ 1\ 2\ 7\ 6\ 9\ 5\ 8\ 7\ 3, 0\ 1\ 2\ 0\ 9\ 6\ 5\ 8\ 7\ 4, 0\ 1\ 2\ 0\ 6\ 7\ 9\ 5\ 8\ 4, 0\ 1\ 4\ 1\ 9\ 8\ 7\ 2\ 5$

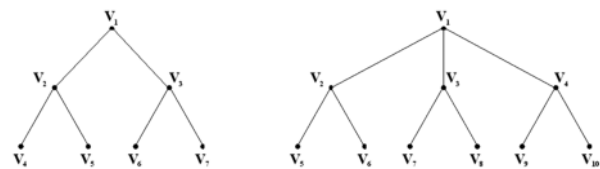


Figure 7: Trees $T(7, 6)$ and $T(10, 9)$

A Doublestar is a Star graph with another layer attached to it as shown in figure 7. Even when tried for a long time for Graceful labelling, no labelling was produced but on repeating a vertex label, it yields several Graceful labelling in a short time and are given below. The vertices are labeled in the order v_1, v_2, v_3, \dots as shown in figure 8.

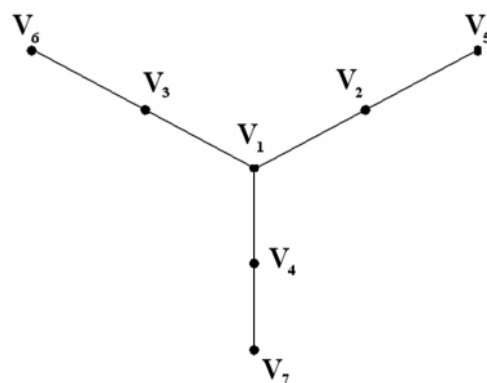


Figure 8: Doublestar

Doublestar (7,6) : 0 1 5 0 3 2 4
 0 2 4 0 3 1 5
 1 3 1 5 6 2 0

Some of the Sequential labellings obtained for Doublestar(7,6) with repeating a vertex label are as follows :

Doublestar(7,6) : 0 4 3 6 1 5 1
 0 4 5 6 3 4 2

3. CONCLUSION

This paper discusses the instances of authors where they have used computer in computing graph labellings and

in particular the results obtained by the computer software developed by Krishnaa(2001) since the results obtained by this software are several. The software developed by Krishnaa(2001) tests the existence of the major labelling schemes namely Harmonious, Graceful, Sequential, Felicitous, Antimagic and Magic for an arbitrary graph and also gives an option to test the case when a single vertex label is repeated for certain labellings. This software makes it possible to test many cases and sheds light on many possibilities which are not otherwise possible manually. This work also discusses some new findings for the complete bipartite graphs $K_{n,n}$ when n is even for Antimagic labellings. Many other possibilities exist which remain to be explored and could enhance the understanding of graph labelling schemes and the use of computer could help us do that.

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